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CONJECTURA PHYSICA

CIRCA PROPAGATIONEM SONI AC LUMINIS

UNA CUM

ALIIS DISSERTATIONIBUS ANALYTICIS

DE NUMERIS AMICABILIBUS

DE NATURA ÆQUATIONUM, AC

DE RECTIFICATIONE ELLIPSIS

AUCTORE

LEONHARDO EULERO.

BEROLINI,

SUMTIBUS AMBR. HAUDE VIDUÆ ET JOH. CAROL. SPENERI,

BIBLIPOL. REG. ET ACAD. SCIENT. PRIVIL.

1750.

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Conjectura Phyſica
de
Propagatione Soni ac Luminis.

§. I.

In motu corporum ac præcipue fluidorum plurima occurrunt phænomena, quæ per Theoriam nondum explicari poſſunt. Etſi enim principia Mechanicæ, a quibus omnes motus determinationes pendent, ſatis cognita atque ad quosvis caſus accommodata videntur, ut eorum ope motus immutationes formulis analyticis includi queant; tamen ſæpenumero ipſa Analyſis his formulis evolvendis impar deprehenditur. His igitur caſibus non tam Mechanica, ad quam ſcientiam motuum inveſtigatio proprie pertinet, imperfectionis

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eſt accuſanda, quam Analyſis, cujus munus in reſolvendis æquationibus, ad quas reliquæ Matheſeos partes perduxerint, præcipue verſatur. Sic quanquam Geometræ poſt inventum calculum infinitorum nimium ſtudii in Analyſi excolenda multis collocaſſe videntur; tamen frequenter a principiis mechanicis ad ejuſmodi æquationes differentiales deducimur, quarum reſolutio, antequam Analyſis multo adhuc majora ceperit incrementa, fruſtra tentatur. Quamobrem eorum labor, qui omnem operam in promovendis Analyſeos finibus impendunt, non ſolum non eſt reprehendendus, ſed etiam quam maxime laudandus.

§. II. Hunc Analyſeos defectum potiſſimum in Aſtronomia Theoretica deprehendimus. Cum enim huic ſcientiæ ſit propoſitum ex viribus, quibus corpora coeleſtia ſe mutuo impellunt, eorum motus determinare, ſi hoc negotium minus ſucceſſerit, a mechanica certe omnis ſuſpicio culpæ removeri debet. Quæcunque enim concipiantur vires, quibus planeta quiſpiam ſollicitetur, motus ejus ſemper per æquationes mere analyticas exprimi poteſt; ita ut accurata motus deſcriptio a reſolutione harum æquationum, in quo Analyſeos officium verſatur, pendeat. At niſi virium illarum lex ſimpliciſſima ſtatuatur, æquationes iſtæ tantopere fiunt implicatæ, ut omnia artificia, quæ in Analyſi adhuc ſunt detecta, ad eas reſolvendas minime ſufficiant. Atque ita Analyſeos maxime defectui eſt tribuendum, quod motus Lunæ, cunctæque ejus inæqualitates nondum ad certas leges revocari, ac tabulis Aſtronomicis comprehendi potuerint.

§. III. Interim tamen ejusmodi quoque ſe offerunt quæſtiones ad quas expediendas non tam Analyſis, quam ipſius Mechanicæ cognitio ſufficiens deſideratur. In ſolidis quidem corporibus hoc potiſſimum evenit, quando circa axes mobiles in gyrum aguntur: hoc enim caſu ea Mechanicæ principia nobis adhuc ſunt occulta, ex quibus variationem hujusmodi motuum definire liceat.

Maxime

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Maxime autem iſte Mechanicæ defectus in motu & agitatione fluidorum cernitur: in quo fere penitus ignoramus, qua ratione ſingulæ fluidi partes in ſe invicem agant, ſuosque inter ſe motus perturbent. Quamobrem etſi curſus fluminum, & fluxus aquarum per canales ſatis commode ad calculum revocari poſſet; tamen motum fluidorum, qui vocatur inteſtinus, quo ſingulæ fere particulæ inter ſe agitantur, nullo adhuc modo certis legibus circumſcribere licuit.

§. IV. Cum igitur ſonus in motu quodam vibratorio, quo minimæ aeris particulæ inter ſe commoventur, conſiſtat; quomodo hic motus ſit comparatus, & qua lege ex aliis aeris particulis in alias transmittatur, accurate explicari adhuc nequit. Neque Newtonus, & qui poſt eum hoc negotium ſunt aggreſſi, ſoni per aerem propagationem ſatis dilucide expoſuiſſe ſunt cenſendi. In diſſertatione enim mea de Lumine & Coloribus, ubi Newtoni doctrinam illuſtravi, luculenter monſtravi, Virum Acutiſſimum propagationem pulſuum per medium elaſticum non ex ſolis mechanicæ principiis determinaviſſe, ſed hypotheſin quanpiam experimentis quidem confirmatam ſubtili admodum modo in ſubſidium vocaſſe; ſicque primarium ſoni phænomenon, quo uniformiter per aerem proferri obſervatur, non tam explicaſſe, ſed potius ei inveſtigationem ſuam ſuperſtruxiſſe, quod tamen ex ſola Theoria demum deduci debuiſſet.

§. V. Tantum abeſt, ut hoc Newtoni inſtitutum, quo cum principiis mechanicis hypotheſes conjunxit, reprehendendum exiſtimem, ut potius hanc viam ob defectum idoneorum principiorum ſolam eſſe arbitrer, quæ nos ad aliquam ſaltem certam cognitionem perducere valeat. Cum enim determinationem celeritatis, qua ſonus per aerem promovetur, ſuſcipit, commodè & quaſi præter opinionem evenit, ut eæ quantitates, quas hypotheſes introduxerant, penitus ex calculo egrediantur, atque concluſio

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ab iſtis hypotheſibus ita immunis obtineatur, ut intelligi poſſit, etiamſi aliæ hypotheſes fuiſſent aſſumtæ, eandem tamen concluſionem prodituram fuiſſe; quo ipſo veritas concluſionis maxime confirmatur. Verum tamen etſi hoc modo celeritas, qua pulſus per aerem aliudve fluidum elaſticum tranſit, recte definiri videtur, tamen hinc ipſa particularum agitatio, qua pulſus continetur, non elicitur: atque ea, quam calculus exhibet, ab hypotheſibus maxime pendet, ideoque pro vera agnoſci nequit.

§. VI. Porro etiam ita in celeritate pulſuum, quae per methodum Newtoni invenitur, acquieſcendum puto, ut etiamſi experientia adverſari videatur, tamen inde Theoriæ nihil plane detrahatur. Invenit autem Newtonus pulſum quemque per aerem tanta celeritate propagari oportere, ut ſingulis minutis ſecundis ſpatium 979 pedum Londinenſium conficiat, cum tamen experimentis conſtet ſonum ſingulis minutis ſecundis 1140 pedes percurrere. Cauſam quidem hujus accelerationis Newtonus in eo ponit, quod aerem plurimis ejusmodi particulis impraegnatum eſſe arbitratur, per quas pulſus in inſtanti propagentur, ita ut ſi aer hujusmodi particulis omnino eſſet repletus, ſonus quoque ſine mora per intervalla quantumvis magna transferretur. Quare ut ipſam ſoni celeritatem obſervatam obtineat, ſeptimam fere aeris partem ejus naturæ aſſumere cogitur, ut per eam pulſus puncto temporis, quaſi per corpuscula perfecte dura Carteſii, tranſirent.

§. VII. Verum hæc explicatio pluribus premitur difficultatibus iisque tam gravibus, ut nullam veriſimilitudinis ſpeciem retineat. Aer enim ſi tanta copia corpuſculorum durorum redundaret, eas proprietates, qua ipſi ineſſe novimus, elaſticitatem, & comprimendi facultatem penitus amitteret. Namque ſi ſeptima pars, vel etiam decima, quam Newtonus aſſumit, hujusmodi particulis eſſet conflata, aer certe non in minus ſpatium quam ſubſeptuplum vel ſubdecuplum comprimi poſſet, ac tum etiam omnem elateris

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elaterìs vim amittere deberet. Cum autem tam per experimenta conſtet, quam ex vi pulveris pyrii colligere liceat, aerem non ſolum in multo minus ſpatium redigi poſſe, ſed etiam tum maxima vi elaſticitatis gaudere, nullus amplius locus ſententiæ Newtonianæ relinquitur, Præterea quoque experimentis compertum eſt, ſonum pari celeritate per aerem transmitti, ſive is majore ſive minore vaporum copia ſit inquinatus; unde evidens eſt, partium heterogenearum, quæ aeri ſunt permixtæ, copiam nihil omnino ad ſonum accelerandum conferre.

§. VIII. Ego vero nequidem hujusmodi ſubterfugio opus eſſe arbitror ad veritatem Theoriæ ſalvandam; neque enim concedendum puto, Theoriam ab experientia diſſentire. Namque caſus, ad quem Theoria eſt accommodata, prorſus discrepat ab eo, qui per experimenta adverſari videtur. Quod quo facilius perſpiciatur, recordandum eſt in Theoria unicum tantum pulſum, qui nullos alios habeat inſequentes, conſiderari, cum per experimenta celeritas ſoni, hoc eſt ingentis pulſuum ſe mutuo inſequentium frequentiæ declaretur. Nusquam autem in theoria probatum eſt, plures pulſus ſe mutuo quaſi e veſtigio inſequentes eadem celeritate per medium elaſticum propagari debere, qua unicus pulſus ſolitarius progreditur: quinpotius in loco ſupra allegato, ubi doctrinam Newtoni explicavi, celeritatem pulſuum ab inſequentibus affici debere animadverti.

§. IX. Cum igitur nullum experimentum proferri poſſit, quo celeritas unici pulſus exhibeatur, propterea quod promtiſſimus ictus, in quo nulla mora ineſſe videatur, ſatis magnam pulſuum ſeriem creat; atque non ſolum demonſtrari nequeat, pulſuum frequentiam nihil in eorum celeritate mutare, ſed etiam probabile videatur, celeritatem ſoni, tam ob agitationem aeri jam inſitam, quam ab impulſu inſequentium pulſuum accelerari debere: nulla omnino pugna inter theoriam & experientiam adhuc concludi poteſt.

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Quare potius e contrario concludere licebit, ſi Theoria vera ſit, unicusque pulſus tantum ſpatium 979. pedum uno minuto ſecundo percurrat; majorem illam ſoni celeritatem, quam experientia oſtendit, nulli alii cauſæ niſi frequentiæ pulſuum tribui debere. Atque hinc generatim affirmare non dubito, quo major ſit pulſuum ſonum quempiam conſtituentium frequentia, eo celerius hunc ſonum per aerem propagari.

§. X. Neque tamen hanc ſententiam tanquam veritatem propono, ſed tantum pro ejusmodi conjectura haberi volo, quæ fortaſſe, poſtquam plura alia phænomena conſuluerimus atque ad examen vocaverimus, ad ſummum certitudinis gradum evehi queat. Quoniam enim ne unius quidem pulſus promotionem per aerem quietum ex ſolis mechanicæ principiis definire licet; multo minus hæc principia ſufficient ad propagationem plurium pulſuum ſe invicem inſequentium determinandam. Interim tamen facile intelligitur, a pulſuum frequentia eorum celeritatem non mediocriter affici debere: primo enim cum aeris particulæ a pulſibus antecedentibus jam ſint in quapiam agitatione conſtitutae, pulſus ſequentes aliam inde accelerationem adipiſci neceſſe eſt. Deinde quia in quolibet pulſu particulæ aeris motu reciproco agitantur, et tam antrorſum quam retrorſum concitantur, neceſſario evenire debet, ut hæc agitatio in particulas antecedentis pulſus vim quandam exerat, quæ eo erit major, quo pulſus ſibi fuerint propiores eorumque propterea frequentia major: hocque ergo caſu pulſus præcedentes ab inſequentibus magis propellentur, ſicque ſoni celeritas augebitur.

§. XI. Quamvis autem hac conſideratione conjectura mea jam ſatis probabilis videatur, tamen experimenta, quorum ingentem numerum Sollertiſſimus Derham omni adhibita cura inſtituit, contrarium nobis perſuadere videntur. Compertum enim eſt omnis generis ſonos, ſive debiles, ſive vehementes ſive, etiam graves ſive acutos

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cutos pari celeritate per aerem propagari. Quod quidem ad ſoni vehementiam ac debilitatem attinet, ex theoria etiam colligitur, hinc in celeritate ſoni nullum diſcrimen oriri poſſe: verum quia ſoni acuti majori pulſuum frequentia conſtant, graves contra minori, ſecundum conjecturam meam ſoni acutiores celerius per idem ſpatium promoveri deberent, quam graviores; quod cum a Derhamo negetur, videndum eſt, an ejus experimentis vel potius concluſionibus, quas inde deduxit, tanta vis tribui queat, qua conjectura mea evertatur: neque enim ſine plena evictione contrarii de ſententia alias probabili decedere decet.

§. XII. Ac primo quidem ſi intervallum, quod ad experimenta inſtituenda deligitur, non fuerit valde magnum, nullum diſcrimen in velocitate ſonorum acutisſimorum et graviſſimorum percipi poterit. Ponamus enim in ſpatio, quod a ſonis uno minuto ſecundo percurritur, dari pro diverſa ſoni indole differentiam 50 pedum, quæ autem reipſa fortaſſe adhuc multo minor exiſtit. Iam ſi is, qui experimentum capit, a loco, ubi ſonus editur, intervallo 10000 pedum ſit remotus, atque ſono gravisſimo celeritas 1000 pedum pro minuto ſecundo tribuatur, acutisſimo autem celeritas 1050 pedum, obſervator exaudiet ſonum graviſſimum poſt 10″, acutiſſimum autem ſemiminuto ſecundo tantum citius. Quod diſcrimen etiamſi ſenſibile videatur, tamen quia ipſum momentum, quo quisque ſonus editur, tam accurate per ſignum indicari nequit, ut nullus plane error ſit metuendus, merito mihi equidem dubitare videor, an experimenta Derhamiana tam ſint exacta, ut iſta quæſtio per ea decidi posſit.

§. XIII. Deinde vero non ſolum in obſervatione momenti, quo ſonus editur, levis quidam error admitti poteſt, ſed etiam in obſervatione ejus momenti, quo ſonus primum exauditur, propterea quod ſatis parvas minuti ſecundi partes diſtinguere non licet, ita ut ob hanc duplicem cauſam error unius ſemiminuti

minuti ſecundi inevitabilis videatur. Tum vero etiam perpendendum eſt, niſi ambo ſoni, gravis et acutus, ſimul edantur, quod quidem inſtitutio experimentorum vix permittit, a diverſa commotione aeris quandam differentiam oriri poſſe; quoniam a ventis propagatio ſoni tam accelerari quam retardari deprehenditur. Imprimis autem animadverti oportet, in diſtantia 10000 pedum, quam aſſumſi, omnis generis ſonos ratione gravis et acuti diſcrepantes, quales inſtrumentis muſicis edi ſolent, exaudiri non poſſe. At ſi hujusmodi experimenta in minoribus diſtantiis inſtituantur, differentia inter ſonorum gravium et acutorum perceptionem adhuc multo minor evadet, omnemque obſervatoris diligentiam effugiet. Neque ergo conjectura, quam propoſui, per experimenta ullam adhuc probabilitatis diminutionem eſt paſſa.

§. XIV. Verum objicietur, Derhamum in multo majoribus diſtantiis etiam experimenta ſua de ſoni velocitate inſtituiſſe, atque adeo tempus, quo tormentorum fragor per ſpatium 60000 pedum propagetur, eſſe dimenſum. Sed in hujusmodi ſonis, quæ a tormentis ac ſclopetis eduntur, tanta non ineſt diverſitas ratione gravis et acuti, ut inde quicquam ſive ad confirmandam ſive ad refellendam conjecturam meam concludi posſit. Videntur autem hi ſoni vehementer graves; ex quo ſpatium 1140 pedum Anglicorum, per quod ſonus ſingulis minutis ſecundis propelli ex his experimentis colligitur, ſonis tantum graviſſimis erit tribuendum: ita ut ſoni acutiores aliquanto majus ſpatium ſingulis minutis ſecundis percurrere ſint exiſtimandi. Quanto autem celerius ſoni acutiores per aerem propagentur quam graviores, nullis fere experimentis definiri poſſe videtur, quod cum a Theoria multo minus ſit expectandum, omnino in dubio relinquitur, etſi alias ipſa conjectura jam ſatis probabilis videatur.

§. XV. Neque etiam haec experimenta eum in finem commemoravi, ut inde quicquam ad conjecturam corroborandam concludi

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cludi poſſe crederem, ſed tantum ut obiectionem, quae graviſſima videbatur, diluerem, atque oſtenderem ex his experimentis, quæ Derhamus ſumma diligentia inſtituit, nullum argumentum contra conjecturam meam peti poſſe. Quod ſi ergo Newtono concedendum putamus, unicum pulſum in aere non ultra 979 pedes uno minuto ſecundo promoveri, necesſe eſt ut majorem illam ſonorum celeritatem frequentiæ pulſuum ſe invicem proſequentium et quaſi propellentium tribuamus. Ex quo et illud agnoſcere cogimur, quo major fuerit pulſuum frequentia eo majorem quoque illam velocitatis accelerationem eſſe oportere, etiamſi fortaſſe differentia in ſonis gravisſimis et acutiſſimis ſit admodum exigua. Nulla enim ratio ſuadet, ut credamus accelerationem ſoni ultra ſpatium illud 979 pedum frequentiæ pulſuum exacte eſſe proportionalem; ſed fieri poteſt, ut cum frequentia multo ſit major, tamen inde vix notabilis acceleratio oriatur.

§. XVI. Interim tamen non puto diſcrimen hoc in velocitate ſonorum plane eſſe inobſervabile; quantumvis enim id ſit parvum, a Muſicæ peritis percipi poſſe videtur. Quodſi enim plurium diverſorum ſonorum concentus, in quo ſinguli ſoni tam exacte juſtis intervallis ſe invicem inſequi debent, ut vel tenuisſimus error aures offendat, e longinquo audiatur, diſcrimen facile animadvertetur, ſi ſoni acutiores unico quaſi inſtante citius ad aures perferrentur, quam graviores. Admisſa autem conjectura vox gravisſima, quæ Baſſus vocatur, reſpectu reliquarum vocum acutiorum aliquanto tardius e longinquo exaudiri deberet, quam ab adſtantibus, hæcque retardatio ab auribus muſicæ adſuetis multo accuratius ſentietur, quam ſi celeritatem cujusque ſoni per exactisſimas temporis menſuras eo, quo Derhamus uſus eſt modo, inveſtigare vellemus. Quodſi ergo hujusmodi diſcrimen pro variis diſtantiis, ex quibus concentus Muſicus auditur, obſervaretur, hoc ipſo conjectura noſtra extra omnem dubitationem collocaretur.

§. XVII. Deinde ſi in quoque ſono celeritas pulſuum antecedentium ab inſequentibus intenditur, in pulſibus ultimis hæc acceleratio nullum amplius locum habebit, iique propterea celeritate naturali propagabuntur, et ſingulis minutis ſecundis ſpatium tantum 979 pedum conficient, dum pulſus primi eodem tempore ſpatium circiter 1140 pedum abſolvunt. Ex quo ſequitur eundem ſonum, quo longius audiatur, eo diutius durare debere: ponamus enim ſonum unico quaſi ictu ſeu puncto temporis abſolvi, ita ut ejus duratio adſtantibus breviſſima videatur; quod ſi jam idem ſonus in diſtantia 10000 pedum audiatur, primus pulſus exaudietur poſt tempus $\frac{10000}{1140}''$, ultimus autem pulſus demum poſt tempus $\frac{10000}{979}''$, ſicque totum tempus, quo hic ſonns percipietur erit $\frac{10000 \cdot 161}{979 \cdot 1140}''$ ſeu $\frac{10000}{6932}''$. Hinc ergo duratio ſoni in diſtantia 10000 pedum prope $1\frac{1}{2}$ ſec. in diſtantia autem 20000 pedum tribus minutis ſecundis protrahetur. Difficile autem erit hanc ſonorum prolongationem diſcernere, quia objecta interpoſita ob tremorem conceptum jam per ſe ſonos protrahere ſolent.

§. XVIII. Multo autem fortiora argumenta ad conjecturam meam confirmandam ſuppeditant phænomena lucis, quæ ſi probe perpendantur, nullum fere amplius dubium relinquent. Cum enim pulſuum propagatio in æthere perinde lumen efficiat, ac ſonus per pulſus in aere propagatos excitatur; ſi oſtendero in æthere celeritatem pulſuum ab eorundem frequentia pendere, nullo modo dubitare licebit, quin in aere etiam pulſuum propagatio ab eorum frequentia acceleretur. Præcipuum autem lucis phænomenon, quo conjectura mea confirmari videtur, in diverſa refractionis ratione verſatur, quam radii diverſorum colorum, dum ex alio medio diaphano in aliud transeunt, ſequi obſervantur. Quia enim probavi colorum varietatem in nulla alia re poni poſſe, niſi in varia pulſuum, quibus quisque color repræſentatur, frequen-

tia

tia, neceſſe eſt ut diverſæ refractionis rationes a diverſa pulſuum frequentia proficiſcantur, radiique verbi gratia rubri ob eam tantum cauſam minorem refractionem pati ſunt cenſendi, quam violacei, quod in illis major minorve pulſuum frequentia inſit, quam in his.

§. XIX. In Theoria autem mea lucis et colorum luculenter oſtendi, ſi radius lucis ex uno medio in aliud tranſit, ſemper eſſe debere ſinum anguli incidentiæ ad ſinum anguli refractionis in eadem ratione, quam tenet celeritas pulſuum in medio priori ad celeritatem eorum in medio poſteriori. Quanquam autem ibi ſum ſuſpicatus frequentiam pulſuum rationem refractionis immutare poſſe, radiosque rubros, dum ex medio rariori in denſius, ubi tardius progrediuntur, tranſeunt, ideo minus refringi putavi, quod pulſus ſequentes, quia in medio denſiori propius ad ſe invicem accederent, refractionem diminuere videbantur; tamen hæc cauſa ceſſaret, ſi radii ex medio denſiori in rarius ingrediantur: neque enim hoc caſu ob iſtam cauſam refractio diminui deberet, quod tamen in hoc caſu æque ac in priori evenire experiontia teſtatur. Atque hanc ob cauſam illam explicationem, cur radii rubri minorem ſemper patiantur refractionem quam cœrulei, penitus rejiciendam eſſe agnoſco; neque jam pulſuum frequentiam quicquam ad refractionem immutandam conferre poſſe arbitror.

§. XX. Statuo igitur, quæcunque ſit pulſuum frequentia, ſi radius lucis ex uno medio diaphano in aliud tranſit, eum ſemper ita refringi axactisſime, ut ſit ſinus anguli incidentiæ ad ſinum anguli refractionis, uti celeritas, qua pulſus in medio priori propagantur, ad eorum celeritatem in medio poſteriori. Quare cum radii rubri aliam patiantur refractionem ac violacei, neceſſe eſt, ut pulſuum, quibus radii rubri conſtituuntur, celeritas alia ratione in tranſitu per diverſa media immutetur, atque celeritas pulſuum radiorum violaceorum. Per quodvis ergo medium alia erit celeritas radio-

ſuum rubrorum, alia radiorum cœruleorum, et cum hi radii tantum ratione frequentiæ pulſuum inter ſe diſcrepent, perſpicuum eſt, celeritatem pulſuum ſimul ab eorum frequentia pendere, ita ut qui radii diverſa conſtent pulſuum frequentia, iidem per quodvis medium diverſa celeritate progrediantur.

§. XXI. Si unicus pulſus conſideretur, qui a nullis inſequentibus acceleretur, ejus celeritatem per quodvis medium elaſticum ſequenti modo determinari comperi. Concipiatur hoc fluidum vaſi incluſum, ex quo ob vim elaſticam per foramen in ſpatium omni materia vacuum erumpat, et notetur celeritas, qua cum effluet, quæ ſit debita altitudini v; qua inventa, erit celeritas, qua unicus pulſus in iſto medio elaſtico progredietur, debita altitudini $\frac{1}{2} v$; eritque ergo hæc celeritas ad illam, qua idem fluidum in vacuum eſſet erupturum, ut $\sqrt{\frac{1}{2}}$ ad 1 ſeu ut 1 ad $\sqrt{2}$; hoc eſt ut latus quadrati ad ſuam diagonalem. Quare cum illa altitudo v ſit directe ut elaſticitas et inverſe ut denſitas medii, ſequitur celeritatem, qua unus pulſus per hoc medium propagabitur, eſſe in ratione ſubduplicata compoſita ex directa elaſticitatis et inverſa denſitatis. Hæcque ratio locum habebit, etiamſi ipſa pulſuum celeritas major eſſet minorve, quam per Theoriam Newtonianam reperitur. Quæcunque enim fere hypotheſis fingatur ad pulſuum promotionem determinandam, eadem ſemper proportio celeritatis pro ratione denſitatis et elaſticitatis medii reſultat.

§. XXII. Inventa autem celeritate, qua unicus pulſus per quodpiam medium propagari debet, ea celeritas, qua radius lucis per idem medium promovetur, ob pulſuum complurium ſucceſſionem major eſt exiſtimanda. Scilicet ſi celeritas unici pulſus exprimatur per p, celeritas radii lucis eo major erit quam p, quo major fuerit pulſuum hunc radium conſtituentium frequentia. Quo hæc frequentia facilius in calculum introduci queat, ſit numerus pulſuum, qui dato tempore veluti uno minuto ſecundo eduntur $= n$,

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frequentiam hoc ipſo numero n indicare licebit: ſeu conſideretur intervallum temporis, quod inter quemlibet pulſum et proxime inſequentem eſt interjectum, quod erit $= \frac{1}{n}''$, atque frequentia pulſuum erit reciproce ut hoc intervallum $1:n$, ideoque directe ut numerus n. Exponamus autem frequentiam littera x, ita ut evaneſcente x frequentia ceſſet, caſusque ad unicum pulſum reducatur.

§. XXIII. Propoſito ergo radio lucis quocunque, cujus pulſuum frequentia ſit $= x$, qua color, quem hic radius repræſentat, exponitur; ſi iſte radius per medium quodpiam diaphanum progrediatur, in quo celeritas unici pulſus ſit $= p$, celeritas qua ipſe radius per hoc medium propagabitur, major erit quam p, atque exprimetur certa quadam functione litterarum p et x. Cujusmodi autem hæc ſit functio, a priori determinare non licet: inde enim plus non liquet, quam hanc functionem ita eſſe comparatam, ut ea fiat $= p$, ſi frequentia x plane evaneſcat, tum vero ut creſcente frequentia x, ea quoque fiat major quam p. Hujusmodi autem functiones innumerabiles imaginari licet, quæ omnes his memoratis proprietatibus ſint præditæ. Sit enim X functio quæcunque ipſius x, quæ evaneſcat poſito $x = 0$; et quæ creſcente x pariter creſcat, atque celeritas radii lucis hujusmodi formulis exprimi poterit: $p + X$; $p(1+X)$; p^{1+X}; &c. tres autem has hypotheſes potiſſimum examinabo.

§. XXIV. Quænam autem harum formularum in natura locum habere queant, ex phænomenis refractionis propius colligere licebit; tranſeat enim radius lucis, cujus frequentia $= x$, in aliud medium, per quod unicus pulſus propagetur celeritate $= q$; atque in hoc medio celeritas radii erit vel $q + X$ vel $q(1+X)$ vel q^{1+X}. Unde dum radius ex priori medio in hoc tranſit, erit ſinus anguli incidentiæ ad ſinum anguli refractionis, vel ut $p + X$

ad $q+X$; vel ut $p(1+X)$ ad $q(1+X)$; vel ut p^{1+X} ad q^{1+X}. In ſecunda ergo hypotheſi foret ratio refractionis ut p ad q; ideoque a pulſuum frequentia non penderet: cum igitur experientia teſtetur, radios qui ratione frequentiæ pulſuum inter ſe diſcrepant, alia quoque lege refringi, hoc ipſo ſecunda hypotheſis evertitur.

§. XXV. Ex prima hypotheſi, quæ rationem refractionis præbet $= p+X : q+X$ ſequitur, quo major fuerit frequentia pulſuum x, eo magis rationem refractionis ad rationem æqualitatis accedere; ſi enim eſſet $X = \infty$, ipſa haberetur ratio æqualitatis. Cum igitur refractio radiorum rubrorum minor ſit quam radiorum violaceorum, ſi hæc hypotheſis locum haberet, ex ea ſequeretur, radios rubros majori pulſuum frequentia conſtare, quam radios violaceos. At quia ſecundum tertiam hypotheſin ratio refractionis $p^{1+X} : q^{1+X}$, eo continuo magis a ratione æqualitatis recedit, quo major fuerit pulſuum frequentia x; ſi ea locum haberet, ſequeretur, radios rubros minori pulſuum frequentia conſtare, quam radios violaceos. Si igitur ex aliis phænomenis pateret, utrum in radiis rubris major pulſuum frequentia inſit an minor, quam in radiis violaceis? ſimul intelligeremus, utra harum duarum hypotheſium veritati magis eſſet conſentanea.

§. XXVI. Quo autem propius ad phænomena hoc examen accommodemus, ſit x frequentia radiorum rubrorum, y frequentia radiorum violaceorum, tum vero ſit Y talis functio ipſius y, qualis X eſt ipſius x. Deinde ſit p celeritas pulſus unici in aere, et q in vitro, quoniam quidem refractio radiorum diverſi coloris ex aere in vitrum ſumma diligentia eſt explorata. Sit porro $1 : m$ ratio refractionis radiorum rubrorum, et $1 : n$ radiorum violaceorum ex aere in vitrum intrantium, ita ut ſecundum experimenta Newtoni ſit $m = \frac{50}{77}$ et $n = \frac{50}{78}$. Erit

Erit ergo per primam hypotheſin $\frac{p+X}{q+X}=\frac{1}{m}=\frac{77}{50}$ & $\frac{p+Y}{q+Y}=\frac{78}{50}=\frac{1}{n}$: per tertiam vero hypotheſin habebitur $\left(\frac{p}{q}\right)^{1+X}=\frac{1}{m}$ & $\left(\frac{p}{q}\right)^{1+Y}=\frac{1}{n}$. Inde elicitur $X=\frac{mp-q}{1-m}=\frac{50p-77q}{27}$ & $Y=\frac{np-q}{1-n}=\frac{50p-78q}{28}$; hinc vero $1+X=l\frac{1}{m}:l\frac{p}{q}$ & $1+Y=l\frac{1}{n}:l\frac{p}{q}$.

§. XXVII. Conſideretur jam aliud quodcunque medium diaphanum, per quod unicus pulſus propagetur celeritate $=r$, ſitque $1:\mu$ ratio refractionis radiorum rubrorum, et $1:\nu$ radiorum violaceorum, qui ex aere in hoc medium ingrediuntur. Per primam ergo hypotheſin erit:

$\frac{p+X}{r+X}=\frac{1}{\mu}$ & $\frac{p+Y}{r+Y}=\frac{1}{\nu}$: Ergo ob $X=\frac{mp-q}{1-m}$ & $Y=\frac{np-q}{1-n}$

habebimus $\frac{q-p}{(1-m)r+mp-q}=\frac{1}{\mu}$ & $\frac{p-q}{(1-n)r+np-q}=\frac{1}{\nu}$: unde elicimus

$r=\frac{\mu p-\mu q-mp+q}{1-m}=\frac{\nu p-\nu q-np+q}{1-n}$; hincque porro æquationem reſultantem per $p-q$ dividendo: $m\nu-\mu n-m+n+\mu-\nu=0$. Quod ſi ergo detur ratio refractionis radiorum rubrorum ex aere in quodvis medium diaphanum, per hanc æquationem aſſignabitur ratio refractionis radiorum violaceorum.

§. XXVIII.

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§. XXVIII. Quoniam novimus diſcrimen inter refractiones radiorum rubrorum et violaceorum eſſe minimum ſit 1 : a ratio refractionis radiorum mediæ naturæ ex aere in vitrum, exiſtente $a = \frac{20}{31}$, ac ponamus $m = a + d$ et $n = a - d$, erit $d = \frac{85}{77 \cdot 78}$, ideoque quantitas valde parva. Deinde pro tranſitu radiorum ex aere in novum iſtud medium diaphanum ſit 1 : α ratio refractionis radiorum mediæ naturæ, ponaturque pariter $\mu = \alpha + \delta$ et $\nu = \alpha - \delta$ ob d et δ quantitates minimas hi valores pro m, n, μ et ν in æquatione ante inventa ſubſtituti dabunt:

$$\alpha d - a\delta - d + \delta = 0, \text{ ideoque } \delta = \frac{1-\alpha}{1-a} d$$

Si hoc medium diaphanum ſit aqua, pro qua radiorum mediæ naturæ ratio refractionis ſtatuitur 4 : 3 ſeu $\alpha = \frac{3}{4}$; erit $\delta = \frac{31}{4 \cdot 11} \cdot \frac{85}{77 \cdot 78} = \frac{31}{44} d$. Sicque diverſitas refractionis pro radiorum diverſa natura minor eſt in tranſitu ex aere in aquam, quam ex aere in vitrum, in ratione 31 ad 44.

§. XXIX. Sin autem tertiam hypotheſin conſulamus pro hac eadem refractione radiorum ex aere in novum hoc medium tranſeuntium, erit $\left(\frac{p}{r}\right)^{1+X} = \frac{1}{\mu}$ & $\left(\frac{p}{r}\right)^{1+Y} = \frac{1}{\nu}$ ideoque $1+X = \frac{l\frac{1}{\mu}}{l\frac{p}{r}} = \frac{l\frac{1}{m}}{l\frac{p}{q}}$ & $1+Y = \frac{l\frac{1}{\nu}}{l\frac{p}{r}} = \frac{l\frac{1}{n}}{l\frac{p}{q}}$ Hinc ergo erit $l\mu : l\nu = lm : ln$. Quare ſi μ detur atque ita exprimatur ut ſit $\mu = m^e$ erit quoque $\nu = n^e$, unde pro tranſitu radiorum ex aere in diverſa media hæc elicitur regula, ut, quotuplicata eſt ratio refractionis radiorum rubrorum in unum medium reſpectu rationis refractionis eo-

run-

rundem radiorum in medium alterum, totuplicata ſit quoque ratio refractionis radiorum violaceorum in medium prius reſpectu rationis in poſterius. Seu hæc concluſio ita facilius concipi poteſt, ut, ſi ratio refractionis ex medio quovis in aliud quodcunque ſit $1 : \mu$ pro radiis rubris et $1 : \nu$ pro radiis violaceis, ſemper ſit $l\mu$ ad $l\nu$ in eadem ratione, quomodocumque etiam illa media ratione refractionis diſcrepent.

§. XXX. Ponamus ut ante $m = a + d$, $n = a - d$, et $\mu = \alpha + \delta$ et $\nu = \alpha - \delta$ eritque, $l(a+d) : l(a-d) = l(\alpha+\delta) : l(\alpha-\delta)$. Vel ſi ſit $a+d = (a-d)^{e}$ erit quoque $\alpha+\delta = (\alpha-\delta)^{e}$. Quo autem hunc facilius valorem ipſius δ elicere queamus, hoc logarithmos in ſeries convertamus, eritque

$$\frac{la + \frac{d}{a} - \frac{dd}{2aa} + \frac{d^3}{3a^3} \,\&c.}{la - \frac{d}{a} - \frac{dd}{2aa} - \frac{d^3}{3a^3} \,\&c.} = \frac{l\alpha + \frac{\delta}{\alpha} - \frac{\delta\delta}{2\alpha\alpha} + \frac{\delta^3}{3\alpha^3}}{l\alpha - \frac{\delta}{\alpha} - \frac{\delta\delta}{2\alpha\alpha} - \frac{\delta^3}{3\alpha^3}}$$

Jam ob d et δ quantitates minimas, erit proxime $\frac{\delta}{\alpha} l a = \frac{d}{a} l\alpha$ ſeu $\delta = \frac{\alpha l\alpha}{a la} d$.

Si pro medio hoc aqua accipiatur, ut ſit $\alpha = \frac{3}{4}$ reperietur $\delta = \frac{937}{1228} d$, at prima hypotheſis dederat $\delta = \frac{31}{44} d$. In fractionibus decimalibus erit ſecundum primam hypoth. $\delta = 0.\ 7045\, d$, atque ſecundum tertiam hypoth. $\delta = 0.\ 7630\, d$; diſcrimen quo hic valor illum excedit eſt $= 0.\ 0585\, d$.

§. XXXI. Perſpicitur ergo, quantumvis hæ duæ hypotheſes, quarum altera radiis rubris majorem pulſuum frequentiam, altera vero minorem tribuit, quam violaceis, inter ſe diſcrepent, tamen eandem fere differentiam inter refractionem radiorum

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rubrorum ac violaceorum ex utraque oriri. Diſcrepantia quidem ſatis eſt parva, et per experimenta difficulter decidi poſſe videtur, utra ad veritatem propius accedat. Quod ſi autem experimenta tanta cura inſtituantur, ut variatio refractionis pro diverſa radiorum natura tam ex aere in vitrum, quam ex aere in aquam exactiſſime inde innoteſcat, non ſolum utra noſtrarum hypotheſium ſit verior, intelligetur: ſed ſi utraque a veritate nonnihil diſcrepare deprehendatur; facile foret novam excogitare hypotheſin, quae phaenomenis refractionis perfecte ſatisfaciat. Qua inventa facilior fortaſſe via reddetur ad theoriam, unde haec diverſa refractionis ratio explicetur, luculentius evolvendam.

§. XXXII. Quae hactenus de diverſa radiorum lucis refrangibilitate tradidi, multo latius patent, atque locum perinde habent, ſive hypotheſis diverſae celeritatis, quam radiis diverſicoloribus in eodem medio tribui, vera ſit ſive falſa. Cum enim radiorum ex alio medio in aliud tranſeuntium refractio non ſolum a diverſitate mediorum pendeat, ſed etiam a colore ſeu natura radii, litterae $p. q. r$ naturam mediorum, quatenus ab ea refractio pendet, exprimant, et litterae x et y naturam radiorum rubrorum et violaceorum, quatenus ab ea refractio afficitur. Ita ſi p exprimat facultatem refractivam aeris, q vitri, et r aquae, tum vero x contineat naturam radiorum rubrorum et y violaceorum, quaecunque demum his litteris quantitates deſignentur, certum eſt ſi radius ruber ex aere in vitrum tranſeat, fore ſinum anguli incidentiae ad ſinum anguli refractionis, uti eſt functio quaepiam litterarum p et x ad functionem ſimilem litterarum q et x.

§. XXXIII. Denotet $\varphi. px$ hanc litterarum p et x functionem, qua refrangibilitas determinatur, ſitque $\varphi. qx$ ſimilis functio litterarum q et x; atque $\varphi. rx$ litterarum r et x. Tum pari modo pro radiis violaceis ſint $\varphi. py$, $\varphi. qy$ et $\varphi. ry$ ſimiles functiones litterarum p et y, q et y, atque r et y. Jam

pro

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pro radiis rubris	Sin. incid. ad ſin. refract.
ex aere in vitrum intrantibus	ut $1 : m$
ex aere in aquam intrantibus	ut $1 : \mu$
pro radiis violaceis	
ex aere in vitrum intrantibus	ut $1 : n$
ex aere in aquam intrantibus	ut $1 : \nu$

eritque refractiones has per functiones exprimendo

$$1 : m = \varphi . px : \varphi\, qx \quad\Big|\quad 1 : \mu = \varphi . px : \varphi . rx$$
$$1 : n = \varphi . qy : \varphi . qy \quad\Big|\quad 1 : \nu = \varphi . qy : \varphi . ry$$

unde habemus:

$$m = \frac{\varphi . qx}{\varphi . px};\ n = \frac{\varphi . qy}{\varphi . py};\ \mu = \frac{\varphi\, rx}{\varphi . px};\ \nu = \frac{\varphi . ry}{\varphi . py}$$

§. XXXIV. Per experimenta autem, quibus diverſa radiorum refrangibilitas ex aere in vitrum eſt inveſtigata, compertum eſt eſſe $m = \frac{50}{77}$, et $n = \frac{50}{78}$ ita ut ſit $m : n = 78 : 77$. Quanta autem pro radiis ex aere in aquam aliudve medium ingredientibus ſit differentia inter litteras μ et ν, ex indole functionis φ definiri debet, quæ etiamſi ſit ignota, tamen hinc oſtendere licet, non eſſe $\mu : \nu = m : n$. Si enim eſſet $\mu : \nu = m : n$ foret $\varphi . rx : \varphi . ry = \varphi . qx : \varphi . qy$, ideoque foret etiam $\varphi . px : \varphi . py = \varphi . qx : \varphi . qy = \varphi . rx : \varphi . ry$: hinc autem ſequeretur fore $\frac{\varphi . qy}{\varphi . py} = \frac{\varphi . qx}{\varphi . px}$ ac per conſequens $n = m$. Quare cum non ſit $n = m$, manifeſtum eſt fieri non poſſe ut, ſit $m : n = \mu : \nu$; ſicque cum ſit $m : n = 78 : 77$; pro radiis ex aere in aquam ingredientibus, non erit $\mu : \nu = 78 : 77$; quæ ratio etiam in nullo alio tranſitu radiorum ex aere in aliud medium quodcunque locum habere poteſt.

§. XXXV. Ex data autem differentia litterarum m et n, quæ ad tranſitum radiorum ex aere in vitrum pertinet, definire poſſe videtur differentia inter litteras μ et ν, quæ tranſitum radiorum ex aere in aquam aliudve medium diaphanum ſpectant, ſi ratio

tio quædam phyſica in ſubſidium vocetur. Concipiatur nempe medium quoddam A, cujus denſitas quam minime ſuperet denſitatem aeris, ſitque ratio refractionis radiorum rubrorum ex aere in hoc medium intrantium $= 1 : M$, radiorum violaceorum vero $= 1 : N$. Tum concipiantur alia media diaphana A^2, A^3, A^4, A^5, &c. quorum denſitates ita ordine creſcant, ut radiorum ex quolibet medio in proxime ſequens eadem ſit refractio, quæ ex aere in medium primum A. Scilicet ſit

Ratio refractionis - -	radiorum rubrorum	radiorum violaceorum
ex aere in medium A	$1 : M$	$1 : N$
ex medio A in medium A^2	$1 : M$	$1 : N$
ex medio A^2 in medium A^3	$1 : M$	$1 : N$
ex medio A^3 in medium A^4	$1 : M$	$1 : N$
&c.		

§. XXXVI. His poſitis manifeſtum eſt, ſi radii ex aere immediate ingrediantur in medium A^2, fore rationem refractionis pro rubris $= 1 : M^2$, et pro violaceis $= 1 : N^2$: ſimili modo ſi radii ex aere immediate transeant in medium A^3, erit ratio refractionis radiorum rubrorum $= 1 : M^3$, et violaceorum $= 1 : N^3$. Generatim ergo ſi medium quoddam concipiatur ſigno A^ξ reſpondens, in quod radii ex aere penetrent, erit ratio refractionis radiorum rubrorum $= 1 : M^\xi$ et radiorum violaceorum $= 1 : N^\xi$. Quoniam igitur exponens ξ omnes omnino numeros complectitur, medium A^ξ ad omnia plane diaphana media repraeſentanda erit aptum. Quare ſi A^ξ ſumatur ad vitrum exhibendum, erit $M^\xi = m$ et $N^\xi = n$, ideoque logarithmis ſumendis $\xi lM = lm$ et $\xi lN = ln$, ita ut ſit $lm : ln = lM : lN$.

§. XXXVII. Simili modo ſi aliud concipiatur medium A^η, quod ratione refractionis cum aqua conveniat, erit radiorum rubrorum ex aere in hoc medium ingredientium ratio refractionis

$= 1 :$

$= 1 : M^x$ et radiorum violaceorum $= 1 : N^x$, ita ut jam ſit $\mu = M^x$ et $\nu = N^x$, ideoque $l\mu : l\nu = lM : lN$. Unde patet ſi in tranſitu ex aere in aquam aliudve medium diaphanum ſit ratio refractionis radiorum rubrorum $= 1 : \mu$ et radiorum violaceorum $= 1 : \nu$, fore ſemper $l\mu : l\nu = lm : ln$, quae eſt eadem proprietas, quam ſupra hypotheſis tertia ſuppeditaverat; ſicque haec hypotheſis prae reliquis omnibus veritati conſentanea videtur. Ex quo ſequitur, ſi v ſit velocitas unici pulſus per medium quoddam diaphanum, atque φ exprimat pulſuum, qui radium lucis conſtituunt, frequentiam, vel functionem ejus quampiam, fore celeritatem hujus radii per iſtud medium $= v^{1+\varphi}$: ſimul vero hinc eſt concludendum, in radiis rubris minimam ineſſe pulſuum frequentiam, in violaceis vero maximam, propterea quod illi minime, hi vero maxime refringuntur.

§. XXXVIII. Si ratio refractionis ex aere in medium quoddam diaphanum C ſit pro radiis rubris $= 1 : \mu$ et pro radiis violaceis $= 1 : \nu$; cum vero aliud medium diaphanum habeatur D, in quod radiorum ex aere incidentium ſit ratio refractionis pro rubris $= 1 : \mathfrak{m}$ et pro violaceis $= 1 : \mathfrak{n}$, erit $l\mu : l\nu = l\mathfrak{m} : l\mathfrak{n} = lm : ln$. Tranſeant jam radii ex medio C in medium D erit ratio refractionis radiorum rubrorum $= \mu : \mathfrak{m}$ et radiorum violaceorum $= \nu : \mathfrak{n}$, atque logarithmus illius rationis erit ad logarithmum hujus rationis, ut $l\frac{\mu}{\mathfrak{m}}$ ad $l\frac{\nu}{\mathfrak{n}}$ hoc eſt ut $l\mu - l\mathfrak{m}$ ad $l\nu - l\mathfrak{n}$, quae ratio redit ad hanc $lm : ln$, ſicque in tranſitu radiorum ex medio quocunque diaphano in aliud quodcunque, ſemper erit logarith. rationis refractionis radiorum rubrorum ad logar. rationis refractionis violaceorum in ratione conſtante.

§. XXXIX. Haec autem ratio conſtans, cum ſit $m = \frac{50}{77}$ et $n = \frac{50}{78}$, erit $= l\frac{50}{77}$ ad $l\frac{50}{78}$, et quia eorundem numerorum

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logarithmi eandem inter ſe tenent rationem, quicunque valor ſubtangenti logarithmicæ tribuatur, ſumendis logarithmis vulgaribus erit hæc ratio conſtans $= 1875207 : 1931246$ ſeu proxime ut 33 ad 34. Quodſi ergo radii ex medio quocunque in aliud medium quodcunque ingrediantur, erit ſemper log. rationis refractionis radiorum rubrorum, ad log. rationis refractionis radiorum violaceorum ut 33 ad 34. Hinc ſi ratio refractionis radiorum rubrorum per experimenta fuerit explorata, quæ ſit ut 1 ad $\mathfrak{M}$, ex ea facile ratio refractionis radiorum violaceorum, quæ ſit ut 1 ad $\mathfrak{N}$, definietur, cum enim ſit $l\frac{1}{\mathfrak{M}} : l\frac{1}{\mathfrak{N}}$ ſeu $l\mathfrak{M} : l\mathfrak{N} = 33 : 34$ erit $l\mathfrak{N} = \frac{34}{33}\, l\mathfrak{M}$, ideoque $\mathfrak{N} = \mathfrak{M}^{34:33} = \mathfrak{M}^{1+\frac{1}{33}}$.

§. XL. Sit ut ſupra (26) x frequentia pulſuum radiorum rubrorum, y frequentia violaceorum, et X, Y functiones illæ harum quantitatum, quarum ratio in definienda celeritate radiorum haberi debet. Sit præterea p celeritas unici pulſus in medio priori, et q celeritas unici pulſus in poſteriori, eritque ratio refractionis radiorum rubrorum $= p^{1+X} : q^{1+X}$, et violaceorum $= p^{1+Y} : q^{1+Y}$: hinc logarithmi harum rationum inter ſe erunt ut 33 ad 34, ſeu erit $1+X : 1+Y = 33 : 34$. Si igitur daretur numerus X, ex eo definiri poſſet numerus Y: vel etiam ſi ratio numerorum X & Y innoteſceret, inde uterque aſſignari poſſet. Sed quia hic nos Theoria deſerit, nihil amplius hinc concludere licet, etiamſi hæc hypotheſis veritati maxime conſentanea videatur.

De

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De Numeris Amicabilibus.

Definitio.

§. I.

Bini Numeri vocantur amicabiles, ſi ita ſint comparati, ut ſumma partium aliquotarum unius æqualis ſit alteri numero, & viciſſim ſumma partium aliquotarum alterius priori numero æquetur.

Sic iſti numeri 220 & 284 ſunt amicabiles; prioris enim 220 partes aliquotæ junctim ſumtæ: 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 faciunt 284: & hujus numeri 284 partes aliquotæ: 1 + 2 + 4 + 71 + 142 producunt priorem numerum 220.

Scholion.

§. II. Stifelius, qui primus hujusmodi numerorum mentionem fecit, caſu hos duos numeros 220 & 284 contemplatus ad hanc ſpeculationem deductus videtur; analyſin enim ineptam exiſtimat, cujus ope plura iſtiusmodi numerorum paria inveniantur. Carteſius vero analyſin ad hoc negotium accommodare eſt conatus, regulamque tradidit, qua tria talium numerorum paria elicuit, neque præter ea Schotenius, qui multum in hac inveſtigatione deſudaſſe videtur, plura eruere valuit. Poſt hæc tempora nemo fere Geometrarum ad hanc quæſtionem magis evolvendam operam impendiſſe reperitur. Cum autem nullum ſit dubium quin analyſis quoque ex hac parte incrementa non contemnenda ſit conſecutura, ſi methodus aperiatur, qua multo plura hujusmodi numerorum paria inveſtigare liceat, haud abs re fore arbitror, ſi methodos quasdam huc ſpectantes, in quas forte incidi, communicavero. In hunc finem autem ſequentia præmittere neceſſe eſt.

Hy-

Hypotheſis.

§. III. Si n denotet numerum quemcunque integrum poſitivum, cujusmodi numeri hic ſemper ſunt intelligendi, omnium ejus diviſorum ſummam hoc ſigno $\int n$ indicabo, ita ut character $\int$ numero cuipiam præfixus ſummam omnium ejusdem numeri diviſorum denotet: ſic erit $\int 6 = 1+2+3+6 = 12$.

Corollarium. I.

§. IV. Quoniam inter diviſores cujusvis numeri hic ipſe numerus refertur, partes aliquotæ autem cenſentur diviſores ipſo numero excepto, manifeſtum eſt ſummam partium aliquotarum numeri n exprimi per $\int n - n$.

Coroll. 2.

§. V. Quoniam numerus primus nullos alios diviſores admittit præter unitatem & ſe ipſum, ſi n ſit numerus primus erit $\int n = 1 + n$. Cum autem caſu $n = 1$ ſit $\int 1 = 1$, patet unitatem non recte numeris primis annumerari.

Lemma. I.

§. VI. Si m & n fuerint numeri inter ſe primi, ut præter unitatem nullum habeant diviſorem communem, tum erit $\int mn = \int m . \int n$, ſeu ſumma diviſorum producti mn æqualis eſt producto ex ſummis diviſorum utriusque numeri m & n.

Productum enim mn primo habet ſingulos diviſores utriusque factoris m & n, tum vero inſuper diviſibile eſt per producta ex ſingulis diviſoribus numeri m in ſingulos diviſores numeri n. Hi vero omnes ipſius mn diviſores junctim prodeunt, ſi $\int m$ per $\int n$ multiplicetur.

Coroll. I.

§. VII. Si numerorum m & n uterque ſit primus, ideoque $\int m = 1 + m$ & $\int n = 1 + n$, erit ſumma diviſorum producti $\int mn =$

(1+m)

$(1+m)(1+n) = 1+m+n+mn$. Si præterea p ſit numerus primus diverſus ab m & n, erit $\int mnp = \int mn . \int p = \int m . \int n . \int p = (1+m)(1+n)(1+p)$ Hincque ſumma diviſorum cujusque numeri, qui eſt productum ex quotcunque numeris primis diverſis, facile aſſignabitur.

Coroll. 2.

§. VIII, Si m, n, & p non quidem ſint numeri primi, ſed tamen ejusmodi, ut præter unitatem nullum habeant diviſorem communem, tum mn & p erunt numeri inter ſe primi, ac propterea $\int mnp = \int mn . \int p$. Cum autem ſit $\int mn = \int m . \int n$: erit $\int mnp = \int m . \int n . \int p$.

Scholion.

§. IX. Niſi factores $m . n . p$ ſint numeri inter ſe primi, ſumma diviſorum producti, prout per lemma indicatur, non eſt juſta. Cum enim ſecundum lemma ſinguli diviſores factorum $m . n . p$ inter diviſores producti mnp referantur, ſi haberent diviſorem communem, is inter diviſores producti bis numeraretur; at dum quæſtio de ſumma diviſorum cujuspiam numeri inſtituitur, nullum diviſorem bis numerare oportet. Hinc ſi m & n ſint numeri primi ac $m = n$, non erit $\int nn = \int n . \int n = (1+n)^2 = 1+2n+nn$, ſed habebitur $\int nn = 1+n+nn$, neque diviſorem n bis poni convenit. Cum igitur per hoc lemma ſumma diviſorum cujusque numeri, qui eſt productum ex quotcunque numeris primis diverſis, recte aſſignetur, reſiduum eſt, ut pro factoribus æqualibus regula tradatur, cujus ope ſumma diviſorum producti definiri queat.

Lemma. 2.

§. X. Si n ſit numerus primus, erit $\int n^2 = 1+n+n^2$, $\int n^3 = 1+n+n^2+n^3$; $\int n^4 = 1+n+n^2+n^3+n^4$, & generatim erit

$$\int n^k = 1+n+n^2+ \cdots +n^k = \frac{n^{k+1}-1}{n-1}.$$

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Coroll. 1.

§. XI. Cum ſit $\int n = 1 + n$, erit $\int n^2 = \int n + n^2$, vel etiam $\int n^2 = 1 + n\int n$. Simili modo erit $\int n^3 = \int n^2 + n^3$, vel etiam $\int n^3 = 1 + n\int n^2$; porroque $\int n^4 = \int n^3 + n^4$ ſeu $\int n^4 = 1 + \int n^3$ & ita porro. Sicque ex cognita ſumma diviſorum cujusque poteſtatis n^k facile ſumma diviſorum poteſtatis ſequentis n^{k+1} aſſignatur, cum ſit $\int n^{k+1} = \int n^k + n^{k+1}$ ſeu $\int n^{k+1} = 1 + n\int n^k$.

Coroll. 2.

§. XII. Quo ſummæ diviſorum faciſius per factores exprimi queant, notandum eſt eſſe $\int n^3 = (1+n)(1+n^2) = (1+n^2)\int n$; $\int n^5 = (1+n^2+n^4)\int n$; $\int n^7 = (1+n^2+n^4+n^6)\int n = (1+n^4)(1+n^2)\int n$: ſicque ſummæ diviſorum poteſtatum imparium ſemper per factores exhiberi poſſunt: at poteſtatum parium ſummæ diviſorum quandoque erunt numeri primi.

Coroll. 3.

§. XIII. Hinc igitur facile tabula condi poterit, qua non ſolum numerorum primorum, ſed etiam poteſtatum ipſorum ſummæ diviſorum exhibeantur. Cujusmodi Tabulam hic adjicere viſum eſt, in qua omnium numerorum primorum millenario non majorum, eorumque poteſtatum ad tertiam usque & altiores pro minoribus numeris ſummæ diviſorum per factores expreſſæ traduntur.

Num-

Num.	Summa Diviſorum.	Numeri	Summ. Diviſorum.	Numeri	Summa Diviſorum.
2	3	3	2_2	11	$2^2. 3$
2^2	7	3^2	13	11_2	7.19
2^3	3. 5	3^3	$2_3. 5$	11^3	$2^3. 3.61$
2^4	31	3^4	11^2	11^4	5.3221
2^5	$3^2. 7$	3^5	$2^2.7.13$	11^5	$2^2.3^2.7.19.37$
2^6	127	3^6	1093	11^6	43.45319
2^7	3.5.17.	3^7	$2^4.5.41$	11^7	$2^4.3.61.7321$
2^8	7.73	3^8	13. 757	11^8	7.19. 1772893
2^9	3.11.31	3^9	$2^2.11^2.61$	11^9	$2^2.3.5.3221.13421$
2^{10}	23.89	3^{10}	23.3851		
2^{11}	$3^2.5.7.13$	3^{11}	$2^3.5.7.13.73$	13	2.7
2^{12}	8191	3^{12}	797161	13^2	3. 61
2^{13}	3 43.127	3^{13}	$2^2. 547. 1093$	13^3	$2^2.5.7.17$
2^{14}	7.31.151	3^{14}	$11^2.13.4561$	13^4	30941
2^{15}	3.5.17.257	3^{15}	$2^5. 5.17.41.193$	13^5	2.3.7.61.157
2^{16}	131071			13^6	5229043
2^{17}	$3^3.7.19.73$	5	2. 3	13^7	$2^3.5.7.17.14281$
2^{18}	524287	5^2	31		
2^{19}	$3.5^2.11.31.41$	5^3	$2^2.3.13$	17	$2. 3^2$
2^{20}	$7^2.127.337$	5^4	11.71	17^2	307
2^{21}	3 23 89.683	5^5	$2.3^3.7.31$	17^3	$2^2.3^2.5. 29$
2^{22}	47. 178481	5^6	19531	17^4	88741
2^{23}	$3^2.5.7.13.17.241$	5^7	$2^3.3.13.313$	17^5	$2. 3^3. 7.13.307$
2^{24}	31.601.1801	5^8	19.31.829	19	$2^2.5$
2^{25}	3.2731.8191	5^9	2.3.11.71.521	19^2	3.127
2^{26}	7.73.262657			19^3	$2^3. 5.181$
2^{27}	3.5.29.43.113.127	7	2^3	19^4	151.911
2^{28}	233.1103.2089	7^2	3.19	19^5	$2^2.3.5.7^3.127$
2^{29}	$3^2.7.11.31.151.331$	7^3	$2^4.5^2$		
2^{30}	2147483647	7^4	2801		
2^{31}	3.5.17.257.65537	7^5	$2^3.3.19.43$	23	$2^3. 3$
2^{32}	7.23 89.599479	7^6	29.4733	23^2	7.79.
2^{33}	3.43691.131071	7^7	$2^5\ 5^2.1201$	23^3	$2^4.3.5.53$
2^{34}	31.71.127.122921	7^8	$3^2.19.37.1063$	23^4	292561
2^{35}	$3^3.5.7.13.19.37.73.109$	7^9	$2^3.11.191.2801$		
2^{36}	223.616318177.	7^{10}	329 54457.		

Num.	Sum Diviſorum	Num.	Sum. Diviſorum	Num	Summa Diviſor.	Num.	Sum Diviſorum.
29	2.3.5	67	2^2.17	109	2.5.11.	163	2^2.41
29^2	13.67	67^2	3.7^2.31	109^2	3.7.571	163^2	3.7.19.67
29^3	2^2.3.5.421	67^3	2^3.5.17.449	109^3	2^2.5.11.13.457	163^3	2^3.5.41.2657
31	2^5	71	2^3.3^2	113	2.3.19	167	2^3.3.7
31^2	3.331	71^2	5113	113^2	13.991	167^2	28057
31^3	2^6.13.37	71^3	2^4.3^2.2521	113^3	2^2.3.5.19.1277	167^3	2^4.3.5.7.2789
37	2.19	73	2.37	127	2^7	173	2.3.29
37^2	3.7.67	73^2	3.1801	127^2	3.5419	173^2	67.449
37^3	2^2.5.2603	73^3	2^2.5.13.37.41	127^3	2^8.5.1613	173^3	2^2.3.5.29.41.73
41	2.3.7	83	2^2.3.7	131	2^2.3.11	179	2^2.3^2.5
41^2	1723	83^2	19.367	131^2	17293	179^2	7.4603
41^3	2^2.3.7.29^2	83^3	2^3.3.5.7.13.53.	131^3	2^3.3.11.8581.	179^3	2^3.3^2.5.37.433
43	2^2.11	89	2.3^2.5	137	2.3.23	181	2.7.13
43^2	3.631	89^2	8011	137^2	7.37.73	181^2	3.79.139
43^3	2^3.5^2.11.37	89^3	2^2.3^2.5.17.233.	137^3	2^2 3.5.23.1877.	181^3	2^2.7.13.16381
47	2^4.3	97	2.7^2	139	2^2.5.7	191	2^6.3
47^2	37.61	97^2	3.3169	139^2	3.13.499	191^2	7.13^2.31
47^3.	2^5.3.5.13.17	97^3	2^2.5.7^2.941	139^3	2^3.5.7.9661.	191^3	2^7.3.17.29.37
53	2.3^3	101	2.3.17	149	2.3.5^2	193	2.97
53^2	7.409	101^2	10303	149^2	7.31.103	193^2	3.7.1783
53^3	2^2.3^3.5.281	101^3	2^2.3.17.5101.	149^3	2^2.3.5^2.11.101.	193^3	2^2.5^3.97.149
59	2^2.3.5	103	2^3.13	151	2^3.19	197	2.3^2.11
59^2	3541	103^2	3.3571	151^2	3.7.1093	197^2	19.2053
59^3	2^3.3.5.1741	103^3	2^4.5.13.1061	151^3	2^4.13.19.877.	197^3	2^2 3^2.5.11.3881
61	2.31	107	2^2.3^3	157	2.79	199	2^3.5^2
61^2	3.13.97	107^2	7.13.127	157^2	3.8269	199^2	3.13267
61^3	2^2.31.1861	107^3	2^3.3^3.5^2.229	157^3	2^2.5^2.17.29.79.	199^3	2^4.5^2.19801

Num.	Summa Diviſor.	Num.	Summa Diviſorum	Num.	Summa Diviſorum	Num.	Summa. Diviſorum
211	$2^2.53$	263	$2^3.3.11$	313	2.157	373	2.11.17
211^2	3.13.31.37	263^2	$7^2.13.109$	313^2	3.181^2	373^2	$3.7^2.13.73$
211^3	$2^3.53.113.197$	263^3	$2^4.3.5.11.6917.$	313^3	$2^2.5.97.101.157.$	373^3	$2^2.5.11.17.13913$
223	$^5.7$	269	$2.3^3.5$	317	2.3.53	379	$2^2.5.19$
223^2	3.16651	269^2	13.37.151	317^2	7.14401	379^2	3.61.787
223^3	$2^6.5.7.4973.$	269^3	$2^2.3^3.5.97.373.$	317^3	$2^2.3.5.13.53.773$	379^3	$2^3.5.19.71821$
227	$2^2.3.19$	271	$2^4.17$	331	$2^2.83$	383	$2^7.3$
227^2	73.709	271^2	3.24571	331^2	3.7.5233	383^2	147073
227^3	$2^3.3.5.19.5153.$	271^3	$2^5.17.36721$	331^3	$2^3.29.83.1889.$	383^3	$2^8.3.5.14669$
229	2.5.23	277	2.139	337	2.13^2	389	2.3.5.13
229^2	3.97.181	277^2	3.7.19.193	337^2	3.43.883	389^2	7.21673
229^3	$2^2.5.13.23.2017$	277^3	$2^2.5.139.7673$	337^3	$2^2.5.13^2.41.277.$	389^3	$2^2.3.5.13.29.2609$
233	$2.3^2.13$	281	2.3.47	347	$2^2.3.29$	397	2.199
233^2	7.7789	281^2	109.727	347^2	7.13.1327	397^2	3.31.1699
233^3	$2^2.3^2.5.13.61.89$	281^3	$2^2.3.13.47.3037$	347^3	$2^3\,3.5.29.12041$	397^3	$2^2.5.199.15761$
239	$2^4.3.5$	283	$2^2.71$	349	$2.5^2.7$	401	2.3.67
239^2	19.3019	283^2	3.73.367	349^2	3.19.2143	401^2	7.23029
239^3	$2^5.3.5.13^4$	283^3	$2^2.5.71.8009$	349^3	$2^2.5^2.7.60901$	401^3	$2^2.3.37.41.53.67$
241	2.11^2	293	$2.3.7^2$	353	2.3.59	409	2.5.41
241^2	3.19441	293^2	86143.	353^2	19.6577	409^2	3.55897
241^3	$2^2.11^2.113.257$	293^3	$2^2.3.5^2.7^2.17.101$	353^3	$2^2.3.5.17.59.733$	409^3	$2^2.5.41.83641$
251	$2^2.3^2.7$	307	$2^2.7.11$	359	$2^3.3^2.5$	419	$2^2.3.5.7$
251^2	43.1471	307^2	3.43.733	359^2	7.37.499	419^2	13.13537
251^3	$2^3.3^2.7.17^2.109$	307^3	$2^3.5^3.7.11.13.29$	359^3	$2^4.3^2.5.13.4957$	419^3	$2^3.3.5.7.41.2141$
257	2.3.43	311	$2^3.3.13$	367	$2^4.23$	421	2.211
257^2	61.1087	311^2	19.5107	367^3	3.13.3463	421^2	3.59221
257^3	$2^2.3.5^2.43.1321$	311^3	$2^4\,3.13.137.353$	367^2	$2^5.5.23.13469$	421^3	$2^2.13.17.211.401$

Num.	Summa Diviſorum	Num.	Summa Diviſor.	Num.	Summa Diviſorum	Num.	Summa Diviſorum
431	$2^4.3^3$	479	$2^5.3.5$	547	$2^2.137$	601	$2.7.43$
431^2	$7.67.397$	479^2	43.5347	547^2	$3.163.613$	601^2	$3.13.9277$
431^3	$2^6.3^3.293.317$	479^3	$2^6.3.5.89.1289.$	547^3	$2^3.5.137.29921.$	601^3	$2^2.7.43.313.577$
433	$2.7.31$	487	$2^3.61$	557	$2.3^2.31$	607	$2^5.19$
433^2	$3.37.1693$	487^2	$3.7.11317$	557^2	$7^2.6343$	607^2	$3.13.9463$
433^3	$2^2.5.7.31.18749$	487^3	$2^4.5.37.61.641$	557^3	$2^2.3^2.5^3.17.31.73.$	607^3	$2^6.5^2.19.7369$
439	$2^3.5.11$	491	$2^2.3.41$	563	$2^2.3.47$	613	2.307
439^2	$3.31^2.67$	491^2	37.6529	563^2	31.10243	613^2	3.125461
439^3	$2^4.5.11.173.557.$	491^3	$2^3.3.41.149.809$	563^3	$2^3.3.5.29.47.1093$	613^3	$2^2.5.53.307.709$
443	$2^2.3.37$	499	$2^2.5^3$	569	$2.3.5.19$	617	$2.3.103$
443^2	7.28099	499^2	$3.7.109^2$	569^2	$7^2.6619$	617^2	$97.3931.$
443^3	$2^3.3.5^4.37.157$	499^3	$2^3.5^3.13.61.157$	569^3	$2^2.3.5.19.161881$	617^3	$2^2.3.5.103.38069$
449	$2.3^2.5^2$	503	$2^3.3^2.7$	571	$2^2.11.13$	619	$2^2.5.31$
449^2	97.2083	503^2	13.19501	571^2	$3.7.103.151$	619^2	$3.19.6733$
449^3	$2^2.3^2.5^2.100801.$	503^3	$2^4.3^2.5.7.25301.$	571^3	$2^3.11.13.163041.$	619^3	$2^3.5.13.31.14737$
457	2.229	509	$2.3.5.17$	577	2.17^2	631	$2^3.79$
457^2	$3.7.9967$	509^2	43.6037	577^2	$3.19.5851$	631^2	$3.307.433$
457^3	$2^2.5^2.229.4177.$	509^3	$2^2.3.5.17.281.461$	577^3	$2^2.5.13^2.17^2.197.$	631^3	$2^4.79.199081$
461	$2.3.7.11$	521	$2.3^2.29$	587	$2^2.3.7^2$	641	$2.3.107$
461^2	373.571	521^2	$31^2.283$	587^2	547.631	641^2	7.58789
461^3	$2^2.3.7.11106261$	521^3	$2^2.3^2.29.135721.$	587^3	$2^3.3.5.7^2.34457.$	641^3	$2^2.3.107.205441$
463	$2^4.29$	523	$2^2.131$	593	$2.3^3.11$	643	$2^2.7.23$
463^2	$3.19.3769$	523^2	$3.13.7027$	593^2	163.2161	643^2	$3.97.1423$
463^3	$2^5.5.13.17.29.97$	523^3	$2^3.5.7.131.1609$	593^3	$2^2.3^3.5^2.11.13.541$	643^3	$2^3.5^2.7.23.8269$
467	$2^2.3^2.13$	541	2.271	599	$2^3.3.5^2$	647	$2^3.3^4$
467^2	19.11503	541^2	$3.7.13963$	599^2	7.51343	647^2	211.1987
467^3	$2^3.3^2.5.13.113.193$	541^3	$2^2.13.271.11257$	599^3	$2^4.3.5^2.17.61.173$	647^3	$2^4.3^4.5.41.1021$

Num.	Summa Diviſorum	Num.	Summa Diviſorum	Num.	Summa Diviſor.	Num.	Summa Diviſorum.
653	2.3.109	719	$2^4.3^2.5$	773	$2.3^2.43$	839	$2^3.3.5.7$
653^2	$7.13^2.19^2$	719^2	487.1063	773^2	598303	839^2	704761
659^3	$2^2.3.5.109.42641$	719^3	$2^5.3^2.5.53.4877$	773^3	$2^2.3^2.5.43.59753$	839^3	$2^4.3.5.7.109.3229$
653	$2^2.3.5.11$	727	$2^3.7.13$	787	$2^2.197$	853	2.7.61
659^2	13.33457	727^2	3.176419	787^2	$3.37^2.151$	853^2	3.43.5647
659^3	$2^3.3.5.11.17.53.241$	727^3	$2^4.5.7.13.73109$	787^3	$2^3.5.197.241.257$	853^3	$2^2.5.7.13.29.61.193$
661	2.331	733	2.367	797	2.3.7.19	857	2.3.11.13
661^2	3.145861	733^2	3.19.9439	797^2	157.4051	857^2	735307
661^3	$2^2.331.218461$	733^3	$2^2.5.13.367.4133$	797^3	$2^2.3.5.7.19.63521$	857^3	$2^2.3.5^2.11.13.37.397$
673	2.337	739	$2^2.5.37$	809	$2.3^4.5$	859	$2^2.5.43$
673^2	3.151201	739^2	3.7.26041	809^2	7.13.19.379	859^2	3.246247
673^3	$2^2.5.337.45293$	739^3	$2^3.5.37.273061$	809^3	$2^2.3^4.5.229.1429$	659^3	$2^2.5.43.137.2693$
677	2.3.113	743	$2^3.3.31$	811	2.7.29	863	$2^5.3^3$
677^2	459007	743^2	552793	811^2	3.31.73.97	863^2	$7^2.15217$
677^3	$2^2.3.5.113.45833$	743^3	$2^4.3.5^2.31.61.181$	811^3	$2^3.7.13.29.41.617$	863^3	$2^6.3^3.5.13.17.337$
683	$2^2.3^2.19$	751	$2^4.47$	821	2.3.137	877	2.439
683^2	7.66739	751^2	3.7.26893	821^2	7.229.421	877^2	3.7.37.991
683^3	$2^3.3^2.5.19.46649$	751^3	$2^5.47.282001$	821^3	$2^2.3.137\ 337021$	877^3	$2^2.5.439.76913$
691	$2^2.173$	757	2.379	823	$2^3.103$	881	$2.3^2.7^2$
691^2	3.19.8389	757^2	3.13.14713	823^2	3.7.43.751	881^2	19.40897
691^3	$2^3.173.193.1237$	757^3	$2^2.5^2.73\ 157.379$	823^3	$2^4\ 5.103.67733$	881^3	$2^2.3^2.7^2.388081$
701	$2.3^3.13$	761	2.3.127	827	$2^2.3^3.23$	883	$2^2.13.17$
701^2	492103	761^2	579883	827^2	684757	883^2	3.260191
701^3	$2^2.3^3.13.17.97.149$	761^3	$2^2.3.17.127.17033$	827^3	$2^3.3^2.5.13.23.5261$	883^3	$2^3.5.13.17.77969$
709	2.5.71	769	2 5.7.11	829	2.5.83	887	$2^3.3.37$
709^2	3.7.23971	769^2	3.31.6367	829^2	3.211.1087	887^2	13.60589
709^3	$2^2.5\ 37.71.6793$	769^3	$2^2\ 5.7.11.71.17393$	829^3	$2^2.5.17^2.29.41.83$	887^3	$2^4.3.5.29.37.2713$

Num.	Summa Diviſorum	Num.	Summa Diviſorum.
907	$2^2.227$	971	$2^2.3^5$
907^2	$3.7.39217$	971^2	$13.79.919$
907^3	$2^3.5^2.227.16453.$	971^3	$2^3.3^5.197.2393$
911	$2^4.3.19$	977	$2.3.163$
911^2	830833	977^2	7.136501
911^3	$2^5.3.19.29.41.349$	977^3	$2^2.3.5.53.163.1801$
919	$2^3.5.23$	983	$2^3.3.41$
919^2	$3.7.13.19.163$	983^2	103.9391
919^3	$2^4.5.23.37.101.113$	983^3	$2^4.3.5.13.41.7433$
929	$2.3.5.31$	991	$2^5.31$
929^2	157.5503	991^2	$3.7.13^2.277$
929^3	$2^2.3.5.31\ 431521.$	991^3	$2^6.31.491041$
937	$2.7.67$	997	2.499
937^2	3.292969	997^2	$3.13.31.823$
937^3	$2^2.5.7.67.87797$	997^3	$2^2.5.499.99401$
941	$2.3.157$		
941^2	811.1093		
941^3	$2^2.3.13.157.34057$		
947	$2^2.3.79$		
947^2	$7.277.463$		
947^3	$2^3.3.5.79.89681$		
953	$2.3^2.53$		
953^2	181.5023		
953^3	$2^2.3^2.5.53.90821$		
967	$2^3.11^2$		
967^2	$3.67.4657$		
967^3	$2^4.5.11^2.13.7193$		

Scho-

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Scholion.

§. XIV. Uſus hujus tabulæ eſt ampliſſimus in quæſtionibus circa diviſores & partes aliquotas verſantibus reſolvendis. Ejus enim ope cujusque numeri propoſiti ſumma diviſorum facili negotio inveniri poteſt, qua reperta, ſi inde ipſe numerus propoſitus auferatur, remanebit ejus ſumma partium aliquotarum. Ex quo ſtatim conſtat, hujus tabulæ ſubſidio numeros amicabiles, quos ſum traditurus, facile explorari poſſe, utrum ſint juſti nec ne? Quemadmodum autem ope hujus tabulæ cujusvis numeri ſumma diviſorum cognoſci poſſit, in ſequenti lemmate explicabo.

Lemma. 3.

§. XV. Propoſito quocunque numero ejus ſumma diviſorum ſequenti modo colligitur.

Cum omnis numerus ſit vel primus vel productum ex primis, reſolvatur numerus propoſitus in ſuos factores primos, & qui inter ſe fuerint æquales, conjunctim exprimantur. Hoc modo numerus propoſitus ſemper ad hujusmodi formam redigetur $m^{\alpha}.n^{\beta}.p^{\gamma}.q^{\delta}$. &c. exiſtentibus m, n, p, q, &c. numeris primis. Poſito ergo numero propoſito $= N$ cum ſit $N, = m^{\alpha}.n^{\beta}.p^{\gamma}.q^{\delta}$ &c. & factores $m^{\alpha}, n^{\beta}, p^{\gamma}, q^{\delta}$ &c. inter ſe primi; erit $ſN = ſm^{\alpha}.ſn^{\beta}.ſp^{\gamma}.ſq^{\delta}$. &c. & valores $ſm^{\alpha}, ſn^{\beta}, ſp^{\gamma}, ſq^{\delta}$ &c. ex tabula adjuncta patebunt.

1. Exempl. *Sit numerus propoſitus* $N = 360$.

Reſoluto hoc numero in ſuos factores primos erit $N = 2^3.3^2.5$, ideoque $ſ360 = ſ2^3.ſ3^2.ſ5 = 3.5.13.2.3$, ob $ſ2^3 = 3.5$; $ſ3^2 = 13$: $ſ5 = 2.3$.

Unde his factoribus ordinatis fiet $ſ360 = 2.3^2.5.13 = 1170$.

2. Exempl. *Explorentur numeri 2620 & 2924 utrum ſint amicabiles nec ne?*

Cum ſit $2620 = 2^2 . 5 . 131$ & $2924 = 2^2 . 17 . 43$, examen ita inſtituetur.

Numeri propoſiti	2620	2924
per factores expreſſi	$2^2 . 5 . 131$	$2^2 . 17 . 43$
ſummæ diviſorum	7. 6.132	7. 18.44
ſive	5544	5544
Summæ partium aliquotarum	2924	2620

Cum igitur ſummæ partium aliquotarum ſint numeris reciproce æquales, patet propoſitos numeros eſſe amicabiles.

Scholion.

§. XVI. His igitur præmiſſis, quæ ad inventionem diviſorum cujusque numeri pertinent, ipſum problema de inveſtigatione numerorum amicabilium aggrediar, atque ſcrutabor, quemadmodum hujusmodi numeros ratione ſummæ diviſorum inter ſe comparatos eſſe oporteat, quo deinceps facilius eorum inventio per regulas poſt tradendas ſuſcipi queat.

Problema generale.

§. XVII. Invenire numeros amicabiles, hoc eſt duos numeros hujus indolis, ut alter æqualis ſit ſummæ partium aliquotarum alterius.

Solutio.

Sint m & n duo hujusmodi numeri amicabiles, & per hypotheſin $\int m$ & $\int n$ ſummæ diviſorum eorumdem. Erit numeri m ſumma partium aliquotarum $= \int m - m$, & numeri n ſumma partium aliquotarum $= \int n - n$. Hinc ex natura numerorum amicabilium naſcentur hæ duæ æquationes:

ſm

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$$\int m - m = n \ \& \int n - n = m$$

ſive $\int m = \int n = m + n$.

Numeri ergo amicabiles m & n primo habere debent eandem ſummam diviſorum, tum vero oportet, ut hæc communis diviſorum ſumma æqualis ſit aggregato ipſorum numerorum $m+n$.

Coroll. 1.

§. XVIII. Problema ergo huc reducitur, ut quærantur duo ejusmodi numeri, qui habeant eandem diviſorum ſummam, hæcque æqualis ſit aggregato ipſorum numerorum.

Coroll. 2.

§. XIX. Ipſa quidem problematis ratio exigit, ut bini numeri quæſiti ſint inter ſe inæquales: ſin autem deſiderentur æquales, ut ſit $m = n$, fiet $\int n = 2n$ & $\int n - n = n$: hujus ſcilicet numeri geminati n ſumma partium aliquotarum ipſi fiet æqualis, quæ eſt proprietas numeri perfecti. Ergo quilibet numerus perfectus repetitus numeros exhibet amicabiles.

Coroll. 3.

§. XX. Sin autem numeri amicabiles m & n, ut natura quæſtionis poſtulat, ſint inæquales, manifeſtum eſt, alterum eſſe redundantem alterum deficientem; ſumma ſcilicet partium aliquotarum alterius ipſo erit major, alterius vero ipſo minor.

Scholion.

§. XXI. Ex hac quidem generali proprietate parum adjumenti conſequimur ad numeros amicabiles inveniendos, eo quod iſta analyſeos ſpecies, cujus ope æquationem $\int m = \int n = m + n$ evolvere liceat, etiamnunc penitus ſit inculta. Ob quem defectum formulas magis particulares contemplari cogimur, ex quarum indole regulas ſpeciales pro inventione numerorum amicabilium de-

rivare liceat; quorſum etiam pertinet regula Carteſiana a Schotenio commemorata. Ac primo quidem, etiamſi non conſtet, utrum dentur numeri amicabiles inter ſe primi nec ne? formulas generales ita reſtringam, ut numeri amicabiles factorem communem obtineant.

Problema Particulare.

§. XXII. *Invenire indolem numerorum amicabilium, qui communem habeant factorem.*

Solutio.

Sit a communis factor numerorum amicabilium, quorum alter ponatur $= am$, alter $= an$; ſint vero tam m & a, quam n & a numeri inter ſe primi, ut utriusque diviſorum ſumma per præcepta data reperiri queat. Cum igitur primo utriusque eadem eſſe debeat diviſorum ſumma, fiet $\int a . \int m = \int a . \int n$, ideoque $\int m = \int n$. Deinde vero neceſſe eſt ut ſit $\int a . \int m$ ſeu $\int a . \int n$ ipſorum numerorum æqualis aggregato $am + an$, unde habetur $\frac{a}{\int a} = \frac{\int m}{m+n} = \frac{\int n}{m+n}$

Poſitis ergo numeris amicabilibus am & an, primo eſſe oportet $\int m = \int n$, tum vero requiritur, ut ſit $a(m+n) = \int a . \int m$.

Coroll. 1.

§. XXIII. Si ergo pro m & n ejusmodi numeri jam fuerint eruti ut ſit $\int m = \int n$: tum numerus a inveſtigari debet, ut ſit $\frac{a}{\int a} = \frac{\int m}{m+n}$, ſeu ex ratione, quam numerus ad ſummam diviſorum ſuorum tenere debet, ipſe numerus a erit inveſtigandus.

Coroll. 2.

§. XXIV. Si factor communis a fuerit datus, quæſtio ad inventionem numerorum m & n reducitur, qui prouti vel primi

vel

vel compoſiti ex duobus pluribusve primis aſſumuntur, quoniam tum diviſorum ſummæ actu exhiberi poſſunt, regulæ ſpeciales ad eos inveniendos tradi poterunt.

Coroll. 3.

§. XXV. Statim autem perſpicitur utrumque numerum *m* & *n* primum eſſe non poſſe: quare caſus ſimpliciſſimus extat, ſi alter primus, alter vero productum ex duobus numeris primis aſſumatur. Tum uterque productum ex duobus, pluribusve numeris primis ſtatui poterit, unde innumeræ regulæ ſpeciales pro inveniendis numeris amicabilibus derivari poterunt.

Scholion.

§. XXVI. Diverſæ ergo numerorum amicabilium formæ, quæ hinc naſcuntur, ſequenti modo repræſentari poterunt. Sit *a* utriusque communis factor, & *p*, *q*, *r*, *s* &c. numeri primi, quorum nullus ſit diviſor communis factoris *a*: atque numerorum amicabilium formæ erunt:

Forma Prima - - -	$\begin{cases} apq \\ ar \end{cases}$
Forma Secunda - - -	$\begin{cases} apq \\ ars \end{cases}$
Forma Tertia - - -	$\begin{cases} apqr \\ as \end{cases}$
Forma Quarta - - -	$\begin{cases} apqr \\ ast \end{cases}$
Forma Quinta - - -	$\begin{cases} apqr \\ astu \end{cases}$
&c.	

Quanquam numerus harum formarum in infinitum augeri po-

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teſt, minime tamen hinc concludere licet, in his formis omnes numeros amicabiles contineri. Primum enim, dum hic litteræ p, q, r, s, t, &c. numeros primos diverſos ſignificant, non veriſimile eſt, nullos dari numeros amicabiles, in quibus non occurrant poteſtates ejusdem numeri primi. Deinde pariter non conſtat, utrum non dentur numeri amicabiles, qui vel nullum habeant factorem communem a, vel in quibus factor hic non prorſus ſit idem: veluti ſi darentur numeri amicabiles hujus formæ $m^{\alpha} P$ & $m^{\beta} Q$, in quibus exponentes α & β eſſent diverſi; quæ forma propterea in ſuperioribus non contineretur, etiamſi P & Q eſſent producta ex meris numeris primis inter ſe diverſis. Ex his perſpicitur quæſtionem de numeris amicabilibus latiſſime patere; eamque ob hoc ipſum tam eſſe difficilem, ut ſolutio completa vix ſit expectanda. Solutionibus igitur particularibus equidem tantum incumbam, & varias methodos aperiam, quarum ope ex formulis traditis plures numeros amicabiles mihi elicere licuit. Quælibet autem forma duplicem mihi ſuſpeditavit methodum, prout factor communis a vel datus aſſumitur, vel ipſe quæritur; haſque methodos in ſequentibus problematibus exponam.

Problema. I.

§. XXVII. *Invenire numeros amicabiles primæ formæ* apq & ar, *ſi factor communis* a *ſit datus.*

Solutio.

Cum p, q, & r ſint numeri primi, atque $\int r = \int p . \int q$ ſeu $r+1 = (p+1)(q+1)$, ponatur $p+1 = x$ & $q+1 = y$, fietque $r = xy - 1$, Ideoque x & y ejusmodi eſſe oportet numeros, ut tam $x-1$, & $y-1$ quam $xy-1$ ſint numeri primi. Deinde ut $a(x-1)(y-1)$ & $a(xy-1)$ ſint numeri amicabiles, oportet ut eorum aggregatum $a(2xy-x-y)$ æquale ſit ſummæ diviſo-

rum

rum alterutrius $xy \int a$: unde nanciſcimur hanc æquationem $xy \int a = 2axy - ax - ay$ ſeu $y = \frac{ax}{(2a - \int a)x - a}$. Sit brevitatis gratia $\frac{a}{2a - \int a} = \frac{b}{c}$, & $\frac{b}{c}$ ſit valor fractionis $\frac{a}{2a - \int a}$ ad minimos terminos reductæ, eritque $y = \frac{bx}{cx - b}$ ſeu $cy = \frac{bcx}{cx - b} = b + \frac{bb}{cx - b}$, unde habebimus $(cx - b)(cy - b) = bb$. Cum igitur $cx - b$ & $cy - b$ ſint factores ipſius bb, quadratum cognitum bb in ejusmodi binos factores reſolvi debet, quorum uterque numero b auctus fiat per c diviſibilis, & quoti x & y inde emergentes ita ſint comparati, ut $x - 1$, $y - 1$, & $xy - 1$ evadant numeri primi. Quæ conditio quoties obtineri poterit, quod quidem pro quovis valore ipſius a aſſumto ſtatim diſpicitur, toties obtinebuntur numeri amicabiles, qui erunt $a(x - 1)(y - 1)$ & $a(xy - 1)$ Q. E. I.

Coroll.

§. XXVIII. Prout igitur pro a alii aliique numeri accipiuntur, unde valores b & c innoteſcant, regulæ emergent particulares, quarum ope numeri amicabiles, ſi qui in eo genere dantur, facile eruentur.

Regula. I.

§. XXIX. Sit factor communis a poteſtas quæcunque binarii, puta $a = 2^n$ erit $\int a = 2^{n+1} - 1$, ideoque $2a - \int a = 1$, unde erit $\frac{a}{2a - \int a} = 2^n$, & propterea $b = 2^n$ & $c = 1$. Hinc oritur $(x - 2^n)(y - 2^n) = 2^{2n}$.

Quare

Quare cum 2^{2n} alios non habeat factores niſi poteſtates binarii, erit:

$$x-2^{n}=2^{n+k}\qquad\qquad x=2^{n+k}+2^{n}$$
$$y-2^{n}=2^{n-k}\quad \text{ſeu} \quad y=2^{n-k}+2^{n}$$

Quocirca diſpiciendum eſt, an ejusmodi valor pro k detur, ut ſequentes tres numeri

$$x-1=2^{n+k}+2^{n}-1$$
$$y-1=2^{n-k}+2^{n}-1$$
$$xy-1=2^{2n+1}+2^{2n+k}+2^{2n-k}-1$$

fiant numeri primi. Quod ſi ſuccedat erunt numeri amicabiles:

$$2^{n}(2^{n+k}+2^{n}-1)(2^{n-k}+2^{n}-1)$$
$$2^{n}(2^{2n+1}+2^{2n+k}+2^{2n-k}-1)$$

Vel ſit $n-k=m$ ſeu $n=m+k$, fietque

$$x-1=2^{m}(2^{2k}+2^{k})-1=q$$
$$y-1=2^{m}(1+2^{k})-1=p$$
$$xy-1=2^{2m}(2^{2k+1}+2^{3k}+2^{k})-1=r$$

qui numeri, quoties fuerint primi, præbebunt numeros amicabiles.

Caſus. I.

§. XXX. Sit $k=1$; & numeri amicabiles obtinebuntur, quoties ſequentes tres numeri fuerint primi:

3.2

$3 \cdot 2^m - 1$; $6 \cdot 2^m - 1$; & $18 \cdot 2^{2m} - 1$

Tum enim poſitis:

$p = 3 \cdot 2^m - 1$; $q = 6 \cdot 2^m - 1$; & $r = 18 \cdot 2^{2m} - 1$

numeri amicabiles erunt: $2^{m+1} pq$ & $2^{m+1} r$, ob $n = m + k = m + 1$. Hæcque eſt regula Carteſii a Schotenio tradita.

Exemplum. 1.

§. XXXI. Sit $m = 1$; eritque

$p = 3 \cdot 2 - 1 =$	5	numerus primus.
$q = 6 \cdot 2 - 1 =$	11	numerus primus.
$r = 18 \cdot 4 - 1 =$	71	numerus primus.

hinc ergo oriuntur numeri amicabiles:

$2^2 \cdot 5 \cdot 11$ & $2^2 \cdot 71$

Sive 220 & 284, qui ſunt minimi omnium, qui exhiberi poſſunt.

Exempl. 2.

§. XXXII. Sit $m = 2$, eritque $2^m = 4$ & $2^{2m} = 16$ atque

$p = 3 \cdot 4 - 1 =$	11	numerus primus.
$q = 6 \cdot 4 - 1 =$	23	numerus primus.
$r = 18 \cdot 16 - 1 =$	287	numerus non-primus.

hincque adeo nulli numeri amicabiles oriuntur.

Exempl. 3.

§. XXXIII. Sit $m = 3$, eritque $2^m = 8$ & $2^{2m} = 64$ atque

$p = 3 \cdot 8 - 1 =$	23	primus
$q = 6 \cdot 8 - 1 =$	47	primus
$r = 18 \cdot 64 - 1 =$	1151	primus.

 Ergo

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Ergo hinc numeri amicabiles erunt:

$2^4 . 23 . 47$ & $2^4 . 1151$
ſive 17296 & 18416.

Exempla ſeqq.

§. 34. Hæc exempla cum ſequentibus, in quibus exponenti m majores valores tribuuntur, commodius uno conſpectu ita repræſentari poterunt.

Sit $m =$	1	2	3	4	5	6	7	8
erit $p =$	5	11	23	47	95*	191	383	767*
$q =$	11	23	47	95*	191	383	767*	1535*
$r =$	71	287*	1151	4607*	18431*	73727	294911	1179647

Ubi numeri non-primi aſteriſcis ſunt notati: unde hinc tantum terni numeri amicabiles obtinentur, nempe

I. $\begin{cases} 2^2 . 5 . 7 \\ 2^2 . 71 \end{cases}$ II. $\begin{cases} 2^4 . 23 . 47 \\ 2^4 . 1151 \end{cases}$ III. $\begin{cases} 2^7 . 191 . 383 \\ 2^7 . 73727 \end{cases}$

Ulterius autem progredi non licet, quoniam valores ipſius r nimis ſiunt magni, quam ut dignoſci poſſit, utrum ſint primi nec ne? Tabulæ namque numerorum primorum adhuc conſtructæ vix ultra 100000 porriguntur.

Caſus. II.

§. XXXV. Sit $k = 2$ & valores litterarum p, q, r, qui debent eſſe primi, erunt:

$$p = 5 . 2^m - 1$$

$$q = 20 . 2^m - 1$$

$$r = 100 . 2^{2m} - 1$$

quo-

quorum cum poſtremus ſemper ſit per ternarium diviſibilis, ob $2^{2m} = 3a+1$ & $r = 300a+99$, hinc nulli numeri amicabiles conſequuntur.

Caſus. III.

§. XXXVI. Ponatur $k = 3$, eritque

$$p = 9 \cdot 2^{m} - 1$$

$$q = 72 \cdot 2^{m} - 1$$

$$r = 648 \cdot 2^{2m} - 1$$

quorum cum nullus neceſſario videatur diviſorem admittere, valores ipſorum p, q, r, ex valoribus ſimplicioribus ipſius m oriundos hic conjunctim repræſentabo.

$m =$	1	2	3	4	5
$p =$	17	35*	71	143*	287*
$q =$	143*	287*	575*	1151	2303*
$r =$	2591	10367*	41471*	165887	663551

Hinc ergo, quoniam ulterius progredi non licet, nulli numeri amicabiles inveniuntur.

Caſus. IV.

§. XXXVII. Ponatur $k = 4$, & ſequentes tres numeri debebunt eſſe primi.

$$p = 17 \cdot 2^{m} - 1$$

$$q = 272 \cdot 2^{m} - 1$$

$$r = 4624 \cdot 2^{2m} - 1$$

Ubi cum r ſemper ſit multiplum ternarii, patet hinc nullos prodire numeros amicabiles.

Caſus. V.

§. XXXVIII. Ponatur $k = 5$, & ſequentes tres numeri debebunt eſſe primi.

$$p = 33 \cdot 2^m - 1$$
$$q = 1056 \cdot 2^m - 1$$
$$r = 34848 \cdot 2^{2m} - 1$$

Ubi ſtatim patet caſum $m = 1$ eſſe inutilem, cum det $p = 65$. Sit ergo $m = 2$, fietque

$p = 131$; $q = 4223^*$, $r = 557567$

ubi cum q non ſit primus, & majores valores pro m ob defectum tabularum numerorum primorum examini ſubjici nequeant, neque hinc etiam novi numeri amicabiles eruuntur. At vero ob eandem rationem majores valores ipſi k tribuere non licet.

Scholion.

§. XXXIX. Quoniam poteſtates binarii pro a poſitæ valorem ipſius c in fractione $\frac{b}{c} = \frac{a}{2a - \int a}$ unitati æqualem reddiderunt, hincque ſolutiones obtinere licuit, alios valores pro a, qui pariter ipſi c valorem $= 1$ inducant, peram. Inter hos autem imprimis ſunt notandi, qui ex hac forma $a = 2^n(2^{n+1} + e)$ naſcuntur, ſi quidem $2^{n+1} + e$ ſit numerus primus; tum enim fit $2a -$

$\int a$

$fa = e+1$, & $\frac{b}{c} = \frac{2^n(2^{n+1}+e)}{e+1}$: ſi igitur $e+1$ ſit diviſor numeratoris $2^n(2^{n+1}+e)$ valor ipſius c fieret itidem $=1$.

Regula. II.

§. XL. Sit factor communis $a = 2^n(2^{n+1}+2^k-1)$, at $2^{n+1}+2^k-1$ numerus primus, erit ob $e+1 = 2^k$, fractio $\frac{c}{b} = \frac{2^n(2^{n+1}+2^k-1)}{2^k} = 2^{n-k}(2^{n+1}+2^k-1)$, ſi quidem non ſit $k > n$. Hac ergo hypotheſi habebimus $b = 2^{n-k}(2^{n+1}+2^k-1)$ & $c=1$. Quadratum ergo bb in duos ejusmodi factores $(x-b)(y-b)$ reſolvendum eſt, ex quibus non ſolum valores numerorum $x-1=p$ & $y-1=q$, ſed etiam $xy-1=r$ fiant numeri primi. Cujusmodi caſus ſi eruere liceat, erunt numeri amicabiles apq & ar. Verum hic notandum eſt eos caſus rejiciendos eſſe, in quibus aliquis numerorum primorum p, q, r prodit diviſor ipſius a, ſeu æqualis $2^{n+1}+2^k-1$, quia a per nullum alium numerum primum eſt diviſibile.

Sit $n-k=m$, ſeu $n=m+k$, erit $a = 2^{m+k}(2^{m+k+1}+2^k-1)$ & $b = 2^m(2^{m+k+1}+2^k-1.)$ Jam quia $2^{m+k+1}+2^k-1$ debet eſſe numerus primus, ponatur $2^{m+k+1}+2^k-1=f$ ſeu $f=$

$2^k(2^{m+1}+1)-1$, ut ſit $a=2^{m+k}f$ & $b=2^m f$; erit $bb = 2^{2m}ff=(x-b)(y-b)$. Nunc ob f numerum primum, numerus $2^{2m}ff$ duplici modo in genere in duos factores reſolvetur. Priori modo fiet $(x-b)(y-b)=2^{m-\alpha}.f.2^{m+\alpha}f$, ideoque

$$x=2^{m-\alpha}f+2^m f; \qquad p=(2^{m-\alpha}+2^m)f-1$$
$$y=2^{m+\alpha}f+2^m f; \qquad q=(2^{m+\alpha}+2^m)f-1$$
$$\& \; r=(2^{2m+1}+2^{2m+\alpha}+2^{2m-\alpha})ff-1$$

qui tres numeri p,q,r debent eſſe primi. Poſteriori modo reſolutio fiet ita:

$$(x-b)(y-b)=2^{m\pm\alpha}\,.\,2^{m\mp\alpha}ff, \text{ unde fit}$$
$$x=2^{m\pm\alpha}+2^m f; \qquad p=2^{m\pm\alpha}+2^m f-1$$
$$y=2^{m\mp\alpha}ff+2^m f; \qquad q=(2^{m\mp\alpha}f+2^m)f-1$$
$$\& \; r=(2^{2m+1}f+2^{2m\pm\alpha}+2^{2m\mp\alpha}ff)f-1$$

& quoties p,q,r hoc modo prodeunt numeri primi, inde oriuntur numeri amicabiles apq & ar.

Caſus. I.

§. XLI. Sit $k=1$, erit $a=2^{m+1}(2^{m+2}+1)$, $b=$ 2m

$2^m(2^{m+2}+1)$ atque $f = 2^{m+2}+1$, qui numerus debet eſſe primus. Cum ergo ſit $(x-b)(y-b) = 2^{2m}ff$, erit

vel	vel
$p = (2^{m-n}+2^m)f-1$	$p = 2^{m\pm n}+2^m f-1$
$q = (2^{m+n}+2^m)f-1$	$q = (2^{m+n}f+2^m)f-1$
$r = (2^{2m+1}+2^{2m+n}+2^{2m-n})ff-1$	$r = (2^{2m+1}f+2^{2m\pm n}+2^{2m+n}ff)f-1$

Notandum autem eſt, ut $2^{m+2}+1$ ſit numerus primus, exponentem $m+2$ eſſe oportere poteſtatem binarii: valores ergo ipſius m erunt: 0, 2, 6, 14, &c. At caſus $m=0$ rejici debet, ob nullum valorem ipſius n aſſignabilem.

Exemplum. I.

§. XLII. Sit ergo $m=2$, ut ſit $a = 8.17$ & $b = 4.17 = 68$ atque $f=17$. Cum igitur eſſe debeat $(x-b)(y-b) = 4^2.17$, erit reſolutione in factores inſtituenda:

$x-68=$	2	4	8	34
$y-68=$	8.17^2	1156	578	136
$x=$	70	72	76	102
$y=$	2380	1224	646	204
$p=$	69*	71	75*	101
$q=$	2379*	1223	645*	203*
$r=$	166599*	88127*	49095*	20807

Hinc ergo nulli numeri amicabiles obtinentur.

Exem-

Exemplum. 2.

§. XLIII. Sit $m=6$, ut $a=2^7.257$; $b=2^6.257$ & $f=257$. Cum igitur ſit

$$(x-b)(y-b)=2^{12}.257^2$$

Reſolutio ita inſtitui debebit:

$x-16448=$	32.257
$y-16448=$	128.257
$x=$	24672
$y=$	49344
$p=$	24671
$q=$	49343*
$r=$	- - - -

Valores ex reliquis factoribus oriundi adhuc magis fiunt magni quam ut, an primi ſint nec ne, judicari poſſit.

Caſus reliqui.

§. XLIV. Cum $f=2^{m+k+1}+2^k-1$ debeat eſſe numerus primus, quæramus primo caſus ſimpliciores, quibus hoc evenit, cum caſus nimis compoſitos evolvere non liceat. Sit ergo $k=2$, & ob $f=2^{m+3}+3$, valores idonei pro m erunt: 1, 3, 4: ſit $k=3$, erit $f=2^{m+4}+7$, & valores idonei pro m erunt 2, 4, 6. Caſu $k=4$, eſt $f=2^{m+5}+15$, & m erit 1, vel 3 neque ulterius progredi licet.

Exempl. 1.

§. XLV. Ponamus ergo $k=2$, & $m=1$, erit $f=19$; & $a=8.19$ atque $b=2.19=38$, unde fiet

(x-

$$(x-38)(y-38) = 2^2 . 19^2 = 1444.$$

& reſolutio dabit.

$x-38 =$	2	4	
$y-38 =$	722	361	Neuter ſcilicet factor aſſumi poteſt impar
$x =$	40		
$y =$	760	imp:	
$p =$	39*		

Quia hic jam p non eſt primus, patet hinc nullos numeros amicabiles reſultare.

Exempl. 2.

§. XLVI. Ponamus $k = 2$ & $m = 3$, ut ſit $f = 67$ erit $a = 32 . 67$ & $b = 8 . 67 = 536$: und ſit

$$(x-536)(y-536) = 2^6 . 67^2$$

$x-536$	268	16	reliqui valores pro p præbent numeros per 3 diviſibiles, quos propterea omiſi. Sequentia exempla ad nimis magnos numeros deducunt.
$y-536$	1072	17956	
$x =$	804	552	
$y =$	1608	- - -	
$p =$	803*	1551*	
$q =$	1607	- - -	

Regula III.

§. XLVII. Sit ut ante $a = 2^n (2^{n+1} + 2^k - 1)$ & $2^{n+1} + 2^k - 1 = f$ numerus primus, at in fractione $\frac{b}{c} = \frac{2^n (2^{n+1} + 2^k - 1)}{2^k}$ ſit $k > n$; eritque $b = 2^{n+1} + 2^k - 1$ & $c = 2^{k-n}$. Ponamus $k - n = m$, ut ſit $k = m + n$, erit $a = 2^n (2^{n+1} + 2^{m+n} - 1)$; $b =$

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$b = 2^{n+1} + 2^{m+n} - 1 = f$ & $c = 2^m$; unde hæc habebitur æquatio

$$(2^m x - b)(2^m y - b) = bb$$

Cum autem $b = f$ ſit numerus primus, alia reſolutio locum non invenit præter $1.bb$: ex qua fit

$$x = \frac{1+b}{2^m} \quad \& \quad y = \frac{b(1+b)}{2^m} \quad \text{ſive}$$

$$x = 2^n + 2^{n+1-m} \quad \& \quad y = (2^{n+1} + 2^{m+n} - 1)(2^n + 2^{n+1-m})$$

Jam notandum eſt hos quatuor numeros eſſe oportere primos:

$$f = 2^{n+1} + 2^{m+n} - 1$$

$$p = x - 1;\ q = y - 1;\ \&\ r = xy - 1$$

atque neceſſe eſt ut ſit $m < n+1$. Quibus conditionibus ſi ſatisfiat, erunt numeri amicabiles: apq & ar.

Caſus. 1.

§. XLVIII. Sit $m = 1$, erit $f = 2^{n+2} - 1$; $x = 2^{n+1}$, & $p = 2^{n+1} - 1$, fieri autem nequit, ut ſimul & f & p ſit numerus primus, niſi caſu $n = 1$, quo vero fit $q = 27$. Ergo ex hypotheſi $m = 1$ nulli oriuntur numeri amicabiles.

Caſus. 2.

§. XLIX. Sit ergo $m = 2$, ut ſit $f = 3.2^{n+1} - 1$; $x = 3.2^{n-1}$ & $y = 3.2^{n-1}(3.2^{n+1} - 1)$. atque $a = 2^n.f$.

Sequen-

Sequentes ergo quatuor numeri debent eſſe primi:

$f = 3.2^{n+1} - 1$; $p = 3.2^{n-1} - 1$; $q = 3.2^{n-1}(3.2^{n+1} - 1) - 1$ & $r = 9.2^{2n-2}(3.2^{n+1} - 1) - 1$, unde formantur hæc exempla;

$n =$	1	2	3	4	5
$f =$	11	23	47	95*	191
$p =$	2	5	11	- -	47
$q =$	32*	137	563	- -	9167*
$r =$	98*	827	6767*	- -	
		valet			

hincque ergo ex $n = 2$, & $a = 4.23$ naſcuntur numeri amicabiles:

$$\begin{cases} 4.\ 23.\ 5.\ 137 \\ 4.\ 23.\ 827 \end{cases}$$

Caſus ceteri.

§. L. Si $m = 3$, iterum vel f vel p fit diviſible per 3, quod idem evenit ſi $m = 5$, vel 7, &c. Sit ergo $m = 4$; erit $f = 9.2^{n+1} - 1$; $x = 9.2^{n-3}$ & $y = 9.2^{n-3}(9.2^{n+1} - 1)$ & $a = 2^n.f$, unde formantur hæc exempla:

$n =$	1	4	5	6
$f =$	35*	287*	575*	1151
$x =$	-	---	---	72
$y =$	-	---	---	82872
$p =$	-	---	---	71
$q =$	-	---	---	82871*
$r =$	-	---	---	- - -

 Ne-

Neque ergo hinc neque ex majoribus valoribus ipſi m tribuendis numeros amicabiles elicere licet.

Regula. IV.

§. LI. Poſſunt etiam aliæ expreſſiones pro factore communi a inveniri, ex quibus fractionis $\frac{b}{c}$ denominator c vel unitati, vel poteſtati binarii fiat æqualis. Fingamus namque $a = 2^n (g-1)(h-1)$, ut ſint $g-1$ & $h-1$ numeri primi; erit $\int a = (2^{n+1}-1) gh = 2^{n+1} gh - gh$; at eſt $2a = 2^{n+1} gh - 2^{n+1} g - 2^{n+1} h + 2^{n+1}$ unde fit

$$2a - \int a = gh - 2^{n+1} g - 2^{n+1} h + 2^{n+1}$$

Ponatur $2a - \int a = d$, erit $gh - 2^{n+1}(g+h) + 2^{n+1} = d$ & $(g-2^{n+1})(h-2^{n+1}) = d - 2^{n+1} + 2^{2n+2}$: unde per reſolutionem in factores ejusmodi valores pro g & h elici debent, ut $g-1$ & $h-1$ fiant numeri primi, eritque tum $a = 2^n (g-1)(h-1)$ & $\frac{b}{c} = \frac{a}{d}$.

I. Ponamus $n = 1$, erit $(g-4)(h-4) = d + 12$; ubi ut $d + 12$ duos obtineat factores pares, ſequentes prodibunt valores:

Sit $d = 4$; erit $(g-4)(h-4) = 16 = 2.8$, unde $g = 6$, $h = 12$;
$a = 2.5.11$ atque $\frac{b}{c} = \frac{2.5.11}{4}$ ergo $b = 5.11$ & $c = 2$.

Sit

Sit $d = 8$; erit $(g-4)(h-4) = 20 = 2.10$, unde $g = 6$, $h = 14$;

$a = 2.5.13$, atque $\frac{b}{c} = \frac{2.5.13}{8}$: ergo $b = 5.13$ & $c = 4$.

Sit $d = 16$; erit $(g-4)(h-4) = 28 = 2.14$; unde $g = 6$, $h = 18$

$a = 2.5.17$, atque $\frac{b}{c} = \frac{2.5.17}{16}$: ergo $b = 5.17$ & $c = 8$.

II. Ponamus $n = 2$, erit $(g-8)(h-8) = d + 56$; atque $a = 4(g-1)(h-1)$, unde ſequentes caſus reſultant:

Sit $d = 4$, erit $(g-8)(h-8) = 60 = 6.10$, unde $g = 14$ & $h = 18$

$a = 4.13.17$, atque $\frac{b}{c} = \frac{4.13.17}{4}$; ergo $b = 13.17$ & $c = 1$

Sit $d = 8$, erit $(g-8)(h-8) = 64 = 4.16$; unde $g = 12$, & $h = 24$

$a = 4.11.23$, atque $\frac{b}{c} = \frac{4.11.23}{8}$: ergo $b = 11.23$ & $c = 2$

Sit $d = 16$, erit $(g-8)(h-8) = 72 = 6.12$; unde $g = 14$, & $h = 20$

$a = 4.13.19$, atque $\frac{b}{c} = \frac{4.13.19}{16}$; ergo $b = 13.19$ & $c = 4$

III. Ponamus $n = 3$, ut ſit $a = 8(g-1)(h-1)$, oportebitque eſſe $(g-16)(h-16) = d + 240$

Sit $d = 4$, erit $(g-16)(h-16) = 244 = 2.122$; unde $g = 18$, $h = 138$; $a = 8.17.137$ & $\frac{b}{c} = \frac{8.17.137}{4}$; ergo $b = 2.17.137$ & $c = 1$

Sit $d = 8$, erit $(g-16)(h-16) = 248 = 2.124$; unde $g = 18$, $h = 140$

$a = 8.17.139$ & $\frac{b}{c} = \frac{8.17.139}{8}$; ergo $b = 17.139$ & $c = 1$

Sit $d = 16$, erit $(g-16)(h-16) = 256 = 4.64$; unde $g = 20$, $h = 80$

$a = 8.19.79$; $\frac{b}{c} = \frac{8.19.79}{16}$; ergo $b = 19.79$ & $c = 2$

Sit iterum $d = 16$; $(g-16)(h-16) = 8.32$; unde $g = 24$ & $h = 49$

$a = 8.23.47$; $\frac{b}{c} = \frac{8.23.47}{16}$ ergo $b = 23.47$ & $c = 2$.

 Sum-

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Sumtis autem hinc valoribus pro a, ſi numeri amicabiles ſtatuantur $a(x-1)(y-1)$ & $a(xy-1)$, ut ſint $x-1$, $y-1$ & $xy-1$ numeri primi, efficiendum eſt ut ſit $(cx-b)(cy-b)=bb$.

Exempl. 1.

§. LII. Sit $a=2.5.11$, erit $b=5.11=55$ & $c=2$, unde fiet $(2x-55)(2y-55)=5^2.11^2$.

$2x-55$	1	5	25
$2y-55$	3025	605	125
x	28	30	40
y	1540	330	90
$x-1$	27*	29	39*
$y-1$	- - -	329*	- -
$xy-1$	- - -	- - -	- -

Hinc ergo nulli obtinentur numeri amicabiles.

Exemplum. 2.

§. LIII. Sit $a=2.5.13$, erit $b=5.13=65$ & $c=4$; unde fit $(4x-65)(4y-65)=5^2.13^2$.

At hic numerus $5^2.13^2$ non reſolvi poteſt in duos factores, qui 65 aucti fiant per 4 diviſibiles: quod idem in valore $a=2.5.17$ uſu venit.

Exempl. 3.

LIV. Sit $a=4.13.17$, erit $b=13.17=221$ & $c=1$ eſſeque oportet $(x-221)(y-221)=13^2.17^2$, unde

$x-221$	13	17	169
$y-221$	3757	- -	289
$x-1$	233	237*	389
$y-1$	3977*	- - -	509
$xy-1$	- - -	- - -	198899

in

in reſolutione ultima ſit $x-1$ & $y-1$ numerus primus, quæſtio ergo huc redit utrum $xy-1 = 198899$ ſit numerus primus nec ne? Etiamſi autem hic numerus terminum 100000 excedat, tamen demonſtrare poſſum eum eſſe primum, unde numeri amicabiles erunt $\begin{cases} 4.\ 13.\ 17.\ 389.\ 509 \\ 4.\ 13.\ 17.\ 198899 \end{cases}$

Scholion.

§. LV. Numerum autem hunc 198899 eſſe primum inde colligo, quod obſervavi eſſe $198899 = 2.\ 47^2 + 441^2$, ita ut 198899 ſit numerus in hac forma $2aa + bb$ contentus. Certum autem eſt, ſi quis numerus unico modo in forma $2aa + bb$ contineatur, tum eum eſſe primum, ſin autem duplici vel pluribus modis ad formam $2a + bb$ redigi queat, tum eſſe compoſitum. Quæſivi ergo utrum a numero hoc 198899 aliud quadratum duplum præter 47^2 ſubtrahi queat, ut reſiduum evadat quadratum, nullumque ſubducto calculo inveni: ex quo tuto concluſi hunc numerum eſſe primum, ideoque numeros inventos eſſe amicabiles. Ex reliquis autem valoribus ipſius a, quos exhibui, nulli reperiuntur numeri amicabiles.

Regula. V.

§. LVI. Poſſunt etiam alii numeri idonei pro a aſſumi, ex quibus numeros amicabiles eruere liceat. Cum autem pro iis regula generalis tradi nequeat, aliquos tantum hic evolvam, ad quorum imitationem non erit difficile alios excogitare.

I. Sit ergo $a = 3^2.\ 5.\ 13$, erit $\int a = 13.\ 6.\ 14$ & ob $2a = 90.\ 13$ & $\int a = 84.\ 13$, erit $2a - \int a = 6.\ 13$ atque $\frac{b}{c} = \frac{a}{2a - \int a} = \frac{3^2.5.13}{6.13} = \frac{15}{2}$ ideoque $b = 15$ & $c = 2$.

II. Sit

II. Sit $a = 3^2 . 7 . 13$ erit $\int a = 13 . 8 . 14 = 16 . 7 . 13$ unde ob $2a = 18 . 7 . 13$, erit $2a - \int a = 2 . 7 . 13$, ideoque $\frac{b}{c} = \frac{3^2 . 7 . 13}{2 . 7 . 13} = \frac{9}{2}$; unde $b = 9$ & $c = 2$.

III. Sit $a = 3^2 . 7^2 . 13$, erit $\int a = 13 . 3 . 19 . 14 = 2 . 3 . 7 . 13 . 19$ & $2a = 4 . 2 . 3 . 7 . 13$, unde $2a - \int a = 4 . 3 . 7 . 13$, ideoque $\frac{b}{c} = \frac{3^2 . 7^2 . 13}{4 . 3 . 7 . 13} = \frac{21}{4}$, ergo $b = 21$ & $c = 4$.

IV. Sit $a = 3^3 . 5$ erit $\int a = 5 . 8 . 6 = 16 . 3 . 5$. Ergo ob $2a = 18 . 3 . 5$ erit $2a - \int a = 2 . 3 . 5$; hincque $\frac{b}{c} = \frac{3^3 . 5}{2 . 3 . 5} = \frac{9}{2}$ & $b = 9$ & $c = 2$.

V. Sit $a = 3^2 . 5 . 13 . 19$, erit $\int a = 13 . 6 . 14 . 20 = 16 . 3 . 5 . 7 . 13$ & ob $2a = 114 . 3 . 5 . 13$ & $\int a = 112 . 3 . 5 . 13$ erit $\frac{b}{c} = \frac{3^2 . 5 . 13 . 19}{2 . 3 . 5 . 13} = \frac{3 . 19}{2}$ & $b = 3 . 19 = 57$ & $c = 2$.

VI. Sit $a = 3^2 . 7^2 . 13 . 19$, erit $\int a = 13 . 3 . 19 . 14 . 20 = 8 . 3 . 5 . 7 . 13 . 19$ & ob $2a = 42 . 3 . 7 . 13 . 19$ erit $\frac{b}{c} = \frac{3^2 . 7^2 . 13 . 19}{2 . 3 . 7 . 13 . 19} = \frac{21}{2}$, unde fit $b = 21$ & $c = 2$.

Poſitis autem numeris amicabilibus $a(x-1)(y-1)$ & $a(xy-1)$ fieri debet $(cx-b)(cy-b) = bb$.

Exem-

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Exemplum. 1.

§. LVII. Sit $b = 15$, $c = 2$; erit $a = 3^2 . 5 . 13$ & ſatisfieri oportet huic æquationi $(2x - 15)(2y - 15) = 225$:

$2x - 15$	1	5	9
$2y - 15$	225	45	25
x	8	10	12
y	120	30	20
$x - 1$	7	9*	11
$y - 1$	119*	- -	19
$xy + 1$	- - -	- -	239

Numeri ergo amicabiles erunt $\left\{ \begin{array}{l} 3^2 . 5 . 13 . 11 . 19 \\ 3^2 . 5 . 13 . 239 \end{array} \right)$

Exempl. 2.

§. LVIII. Sit $b = 9$, $c = 2$; erit vel $a = 3^2 . 7 . 13$ vel $a = 3^2 . 5$; & æquatio reſolvenda $(2x - 9)(2y - 9) = 81$.

$2x - 9$	3
$2y - 9$	27
x	6
y	18
$x - 1$	5
$y - 1$	17
$xy - 1$	107

Unde cum ſit $x - 1 = 5$, hic valor cum $a = 3^2 . 5$ combinari nequit. Erunt ergo numeri amicabiles:

$\left\{ \begin{array}{l} 3^2 . 7 . 13 . 5 . 17 \\ 3^2 . 7 . 13 . 107 \end{array} \right)$

Exempl. 3.

§. LIX. Sit $b = 21$ & $c = 4$, erit $a = 3^2 . 7^2 . 13$ & æquatio reſolvenda $(4x - 21)(4y - 21) = 441$.

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$4x - 21$	3
$4y - 21$	147
x	6
y	42
$x - 1$	5
$y - 1$	41
$xy - 1$	251

Quia x & y debent eſſe numeri pares alia reſolutio locum non habet.

Ex hac ergo prodeunt numeri amicabiles hi: $\left\{ \begin{matrix} 3^2 . 7^2 . 13 . 5 . 41 \\ 3^2 . 7^2 . 13 . 251 \end{matrix} \right)$

Exempl. 4.

§. LX. Sit $b = 21$ & $c = 2$, erit $a = 3^2 . 7^2 . 13 . 19$ & æquatio reſolvenda $(2x - 21)(2y - 21) = 441$.

$2x - 21$	3	7
$2y - 21$	147	63
x	12	12
y	84	42
$x - 1$	11	13
$y - 1$	83	41
$xy - 1$	1007*	587

Quia autem valor $x - 1 = 13$ jam in valore a continetur, hinc nulli obtinentur numeri amicabiles.

Exemplum. 5.

§. LXI. Sit $b = 57$ & $c = 2$, erit $a = 3 . 5 . 13 . 19$, & æquatio reſolvenda... $(2x - 57)(2y - 57) = 3249$.

$2x - 57$	3	19
$2y - 57$	1083	171
x	30	38
y	570	114
$x - 1$	29	34
$y - 1$	569	113
$xy - 1$	17099	4331*

Hinc ergo oriuntur numeri amicabiles hi:

$$\left\{ \begin{matrix} 3^2 . 5 . 13 . 19 . 29 . 569 \\ 3^2 . 5 . 13 . 19 . 17099 \end{matrix} \right)$$

Exem-

Exemplum. 6.

§. LXII. Sit $b = 45$ & $c = 2$, erit $a = 3^4. 5. 11$, & æquatio reſolvenda. $(2x - 45)(2y - 45) = 2025$

$2x - 45$	3	15
$2y - 45$	675	135
x	24	30
y	360	90
$x - 1$	23	29
$y - 1$	359	89
$xy - 1$	8639*	2699

Hinc ergo oriuntur numeri amicabiles:

$$\left\{ \begin{array}{l} 3^4. 5. 11. 29. 89 \\ 3^4. 5. 11. 2699 \end{array} \right)$$

Exempl. 7.

§. LXIII. Sit $b = 77$ & $c = 2$, erit $a = 3^2. 7^2. 11. 13$, & æquatio reſolvenda $(2x - 77)(2y - 77) = 49. 121$.

$2x - 77$	7	11
$2y - 77$	847	539
x	42	44
y	462	308
$x - 1$	41	43
$y - 1$	461	307
$xy - 1$	19403	13551*

Hinc ergo oriuntur numeri amicabiles:

$$\left\{ \begin{array}{l} 3^2. 7^2. 11. 13. 41. 461 \\ 3^2. 7^2. 11. 13. 19403 \end{array} \right)$$

Exempl. 8.

§. LXIV. Sit $b = 105$, $c = 2$, erit $a = 3^2. 5. 7$. & æquatio reſolvenda $(2x - 105)(2y - 105) = 105^2$.

$2x-105$	3	7	15	35
$2y-105$	3675	--	735	--
x	54	56	60	70
y	1890	--	420	--
$x-1$	53	55*	59	69*
$y-1$	1889	--	419	
$xy-1$	102059	--	25199*	

Cum 102059 ſit numerus primus, quia continetur in forma $8d+3$ & unico modo ad formam $2aa+bb$ reducitur, numeri amicabiles hinc orti erunt

$$\begin{Bmatrix} 3^2.5.7.53.1889 \\ 3^2.5.7.\ 102059 \end{Bmatrix}$$

Scholion.

§. LXV. Numeri ergo amicabiles, quos hactenus ex forma *apq*; *ar* invenimus, ſunt:

I. $\begin{Bmatrix} 2^2.5.11 \\ 2^2.71 \end{Bmatrix}$

II. $\begin{Bmatrix} 2^4.23.47 \\ 2^4.1151 \end{Bmatrix}$

III. $\begin{Bmatrix} 2^7.191.393 \\ 2^7.\ 73727 \end{Bmatrix}$

IV. $\begin{Bmatrix} 4.23.5.137 \\ 4.23.\ 827 \end{Bmatrix}$

V. $\begin{Bmatrix} 4.13.17.389.509 \\ 4.13.17.\ 198899 \end{Bmatrix}$

VI. $\begin{Bmatrix} 3^2.5.13.11.19 \\ 3^2.5.13.\ 239 \end{Bmatrix}$

VII. $\begin{Bmatrix} 3^2.7.13.5.17 \\ 3^2.7.13.\ 107 \end{Bmatrix}$

VIII. $\begin{Bmatrix} 3^2.7^2.13.5.41 \\ 3^2.7^2.13.251 \end{Bmatrix}$

IX.

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IX.	$\begin{cases} 3^2.5.13.19.29.569 \\ 3^2.5.13.19.17099 \end{cases}$
X.	$\begin{cases} 3^4.5.11.29.89 \\ 3^4.5.11.2699 \end{cases}$
XI.	$\begin{cases} 3^2.7^2.11.13.41.461 \\ 3^2.7^2.11.13.19403 \end{cases}$
XII.	$\begin{cases} 3^2.5.7.53.1889 \\ 3^2.5.7.102059 \end{cases}$

Problema 2.

§. LXVI. *Invenire numeros amicabiles ſecundæ formæ* apq ars; *poſitis* p, q, r, s *numeris primis & factore communi* a *dato.*

Solutio.

Cum factor communis a detur, quæratur ex eo valor fractionis $\frac{b}{c} = \frac{a}{2a - \int a}$ in minimis terminis, hincque erit $a : \int a = b : 2b - c$. Deinde cum eſſe debeat $\int p . \int q = \int r . \int s$ ſeu $(p+1)(q+1) = (r+1)(s+1)$, ponatur uterque valor $= \alpha \beta x y$, & ſumatur:

$$p = \alpha x - 1;\; q = \beta y - 1;\; r = \beta x - 1;\; s = \alpha y - 1.$$

Ubi manifeſtum eſt hos numeros α, β, x, y, ejusmodi eſſe debere, ut p, q, r, s fiant numeri primi, & numeri amicabiles erunt

$$a(\alpha x - 1)(\beta y - 1) \;\&\; a(\beta x - 1)(\alpha y - 1)$$

Præterea vero ex natura numerorum amicabilium eſſe debet:

$$\alpha \beta x y \int a = a(\alpha x - 1)(\beta y - 1) + a(\beta x - 1)(\alpha y - 1)$$

ſeu ob $\int a : a = 2b - c : b$ erit

$$\left.\begin{array}{l}2b\alpha\beta xy\\ -\alpha\beta xy\end{array}\right\} = 2b\alpha\beta xy - b\alpha x - b\beta y + 2b$$
$$\qquad\qquad\qquad\qquad - b\beta x - b\alpha y.$$

vel $c\alpha\beta xy = b(\alpha+\beta)(x+y) - 2b$. Unde fit

$$cc\alpha^2\beta^2 xy \begin{array}{l} - bc\alpha\beta(\alpha+\beta)x\\ - bc\alpha\beta(\alpha+\beta)y\end{array} + bb(\alpha+\beta)^2 = -2bc\alpha\beta + bb(\alpha+\beta)^2$$

Quare ſatisfieri debet huic æquationi:

$$(c\alpha\beta x - b(\alpha+\beta))(c\alpha\beta y - b(\alpha+\beta)) = bb(\alpha+\beta)^2 - 2bc\alpha\beta$$

Numerus ergo $bb(\alpha+\beta)^2 - 2bc\alpha\beta$ quovis caſu in duo ejus modi factores, qui ſint PQ, reſolvi debet, ut poſitis

$$x = \frac{P + b(\alpha+\beta)}{c\alpha\beta} \quad \& \quad y = \frac{Q + b(\alpha+\beta)}{c\alpha\beta}$$

hi numeri x & y non ſolum fiant integri, ſed etiam $\alpha x - 1$; $\beta y - 1$; $\beta x - 1$; & $\alpha y - 1$ numeri primi. Erit igitur

$$p = \frac{P + b\alpha + (b-c)\beta}{c\beta};\quad q = \frac{Q + b\beta + (b-c)\alpha}{c\alpha}$$

$$r = \frac{P + b\beta + (b-c)\alpha}{c\beta};\quad s = \frac{Q + b\alpha + (b-c)\beta}{c\alpha}$$

Quovis ergo valore ipſius a propoſito, unde reperitur $\frac{b}{c} = \frac{a}{2a - \int a}$, diſpiciendum eſt, utrum cum numeri α & β ita aſſumi, tum reſolutio hæc:

$$bb(\alpha+\beta)^2 - 2bc\alpha\beta = PQ$$

ita inſtitui queat, ut valores modo traditi pro p, q, r & s fiant numeri primi, & tales quidem, ut factor communis a nullum eorum involvat. Quoties autem his conditionibus ſatisfieri poterit, erunt numeri amicabiles: apq & ars.

Co-

Coroll.

§. LVII. Quoniam eſſe nequit $a = c$, pro his numeris a & c ponantur numeri ſimpliciores, hincque orientur caſus ſequentes:

I. Sit $a = 1$; $c = 2$; erit $PQ = 9bb - 4bc$; &

$$p = \frac{P + 3b - 2c}{2c}; \quad q = \frac{Q + 3b - c}{c}$$

$$r = \frac{P + 3b - c}{c}; \quad s = \frac{Q + 3b - 2c}{2c}$$

II. Sit $a = 1$; $c = 3$; erit $PQ = 16bb - 6bc$ &

$$p = \frac{P + 4b - 3c}{3c}; \quad q = \frac{Q + 4b - c}{c}$$

$$r = \frac{P + 4b - c}{c}; \quad s = \frac{Q + 4b - 3c}{3c}$$

III. Sit $a = 2$; $c = 3$; erit $PQ = 25bb - 12bc$

$$p = \frac{P + 5b}{3c} - 1; \quad q = \frac{Q + 5b}{2c} - 1$$

$$r = \frac{P + 5b}{2c} - 1; \quad s = \frac{Q + 5b}{3c} - 1$$

IV. Sit $a = 1$; $c = 4$; erit $PQ = 25bb - 8bc$ &

$$p = \frac{P + 5b}{4c} - 1; \quad q = \frac{Q + 5b}{c} - 1$$

$$r = \frac{P + 5b}{c} - 1; \quad s = \frac{Q + 5b}{4c} - 1$$

V. Sit

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V. Sit $a = 3$; $c = 4$; erit $PQ = 49\,bb - 24\,bc$

$$p = \frac{P+7b}{4c} - 1; \quad q = \frac{Q+7b}{3c} - 1$$

$$r = \frac{P+7b}{3c} - 1; \quad r = \frac{Q+7b}{4c} - 1$$

VI. Sit $a = 1$; $c = 5$; erit $PQ = 36\,bb - 10\,bc$

$$p = \frac{P+6b}{5c} - 1; \quad q = \frac{Q+6b}{c} - 1; \quad r = \frac{P+6b}{c} - 1; \quad s = \frac{Q+6b}{5c} - 1$$

VII. Sit $a = 2$; $c = 5$; erit $PQ = 49\,bb - 20\,bc$

$$p = \frac{P+7b}{5c} - 1; \quad q = \frac{Q+7b}{2c} - 1; \quad r = \frac{P+7b}{2c} - 1; \quad s = \frac{Q+7b}{5c} - 1$$

VIII. Sit $a = 3$; $c = 5$; erit $PQ = 64\,bb - 30\,bc$

$$p = \frac{P+8b}{5c} - 1; \quad q = \frac{Q+8b}{3c} - 1; \quad r = \frac{P+8b}{3c} - 1;$$

$$s = \frac{Q+8b}{5c} - 1$$

IX. Sit $a = 4$; $c = 5$; erit $PQ.\ 81\,bb - 40\,bc$

$$p = \frac{P+9b}{5c} - 1; \quad q = \frac{Q+9b}{4c} - 1; \quad r = \frac{P+9b}{4c} - 1;$$

$$s = \frac{Q+9b}{5c} - 1$$

X. Sit

X. Sit $a = 1$; $c = 6$; erit $PQ = 49bb - 12bc$.

$$p = \frac{P+7b}{6c} - 1;\ q = \frac{Q+7b}{c} - 1;\ r = \frac{P+7b}{c} - 1;$$

$$s = \frac{Q+7b}{6c} - 1;$$

XI. Sit $a = 5$; $c = 6$, erit $PQ = 121bb - 60bc$

$$p = \frac{P+11b}{6c} - 1;\ q = \frac{Q+11b}{5c} - 1;\ r = \frac{P+11b}{5c} - 1;$$

$$s = \frac{Q+11b}{6c} - 1;$$

Secundum hos igitur caſus valores ipſius a jam ante adhibitos, quia præ ceteris ad numeros amicabiles inveniendos videntur apti, evolvam, ex iis autem potiſſimum eos eligam, qui actu ad numeros amicabiles deducunt.

Exemplum. I.

§. LXVIII. Sit $a = 2^2$; erit $b = 4$, & $c = 1$. Sumatur caſus ſecundus quo $a = 1$, $c = 3$, ut numeri amicabiles ſint 2^2pq & 2^2rs, fierique debet

$PQ = 16.16 - 6.4 = 232$, atque

$$p = \frac{P+16}{3} - 1;\ q = Q + 16 - 1;\ r = P + 16 - 1\ \&\ s = \frac{Q+16}{3} - 1$$

factores ergo numeri 232 ita debent eſſe comparati, ut 16 aucti fiant per 3 diviſibiles:

$P = 2$ Alia reſolutio nulla ſuccedit, ſi enim poneretur
$Q = 116$ $P = 8$, fieret Q numerus impar, neque ergo q

$P + 16 =$	18
$Q + 16 =$	132
$p =$	5
$q =$	131
$r =$	17
$s =$	43

& s numeri primi eſſe poſſent. Hinc ergo obtinentur hi numeri amicabiles:

$$\left\{ \begin{matrix} 2^2.\ 5.\ 131 \\ 2^2.\ 17.\ 43 \end{matrix} \right\}.$$

Exemplum. 2.

§. LXIX. Si $\alpha = 1$ & $\beta = 3$, & a poteſtas binarii altior: inventio numerorum amicabilium non ſuccedit, donec perveniatur ad $a = 2^8$. Tum autem erit $b = 2^8$ & $c = 1$: atque

$$PQ = 16.\ 2^{16} - 6.\ 2^8 = 2^9 (2^{11} - 3) = 512.\ 2045 = 512.\ 5.\ 409;$$

$$p = \frac{P + 1024}{3} - 1;\ q = Q + 1024 - 1;\ r = P + 1024 - 1;\ s = \frac{Q + 1024}{3} - 1$$

unde factores P & Q ita debent eſſe comparati, ut quaternario aucti per 3, (vel ut quoti fiant pares) per 6 ſint diviſibiles.

$P =$	2	8	20	32	80	128	320	1280
$Q =$	- -				13088	8180		
$P + 1024 =$	1026	1032	1044	1056	1104	1152	1344	2304
$Q + 1024 =$	- -	- -			14112	9204		
$p =$	341	343*	347		367	383	447*	767*
$q =$	- - -	- -			14111*	9203		
$r =$	1025*	- -	1043*	1055*	1103	1151	1343	2303
$s =$	- - -	- -	- - -	- -	4703	3067		

Erunt ergo numeri amicabiles $\left\{ \begin{matrix} 2^8.\ 383.\ 9203 \\ 2^8.\ 1151.\ 3067 \end{matrix} \right.$

Exem-

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Exempl. 3.

§. LXX. Sit $\alpha = 2$ & $\varepsilon = 3$ & ſumatur $a = 3^2.5.13$ ut ſit $b = 15$ & $c = 2$; erit $PQ = 25.225 - 12.30 = 3^4.5.13$

$$p = \frac{P+75}{6} - 1;\; q = \frac{Q+75}{4} - 1;\; r = \frac{P+75}{4} - 1;$$

$$s = \frac{Q+75}{6} - 1$$

unde factores P Q ejusmodi eſſe debent, ut ternario aucti fiant per 24 diviſibiles.

P	=	45
Q	=	117
P + 75	=	120
Q + 75	=	192
p	=	19
q	=	47
r	=	29
s	=	31

Aliæ reſolutiones non inveniunt locum; unde hinc numeri amicabiles prodeunt.

$$\begin{cases} 3^2.5.13.19.47 \\ 3^2.5.13.29.31 \end{cases}$$

Exempl. 3.

§. LXXI. Sit $\alpha = 1$ & $\varepsilon = 4$, ſumatur $a = 3^2.5$, ut ſit $b = 9$, $c = 2$, erit $PQ = 25.81 - 8.18 = 9.11.19$ &

$$p = \frac{P+45}{8} - 1;\; q = \frac{Q+45}{2} - 1;\; r = \frac{P+45}{2} - 1;$$

$$s = \frac{Q+45}{8} - 1$$

unde P & Q ejusmodi debent eſſe numeri, ut quinario aucti per 8 fiant diviſibiles:

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P	=	3	19
Q	=	627	99
P + 45	=	28	64
Q + 45	=	672	144
p	=	5	7
q	=	335*	71
r	=	23	31
s	=	83	17

Hinc ergo oriuntur numeri amicabiles:

$$\left\{ \begin{array}{l} 3^3 . 5 . 7 . 71 \\ 3^3 . 5 . 31 . 17 \end{array} \right\}.$$

Scholion.

§. LXXII. Hæ autem operationes nimis ſunt incertæ, ac plerumque plures fruſtra inſtituuntur, antequam numeri amicabiles ſe offerunt. Labor quoque foret vehementer prolixus, ſi ſingulis valoribus ipſius σ, quos quidem ſupra exhibui, per ſingulos caſus litterarum α & β percurrere velimus; atque nimis raro evenit, ut quatuor numeri pro p, q, r & s reſultantes ſimul fiant primi. Tum vero etiam inventio numerorum amicabilium per determinationem rationis α & β nimis reſtringitur, atque dantur caſus hujusmodi numerorum, in quibus ratio α & β tam eſt complicata, ut nulla probabili ratione eligi potuiſſet, cujusmodi ſunt numeri amicabiles $2^4 . 19 . 8563$ & $2^4 . 83 . 2039$, ad quos hac via inveniendos ratio $\alpha : \beta$ aſſumi debuiſſet $5 : 21$ vel $1 : 102$. Hanc ob rem huic methodo nimis ſterili & operoſæ diutius non immoror, ſed aliam viam aperiam, qua facilius & expeditius numeros amicabiles tam hujus ſecundæ formæ, quam aliarum magis compoſitarum inveſtigare liceat; & quæ ſimilis ſit præcedenti, quæ tribus tantum numeris primis reperiendis abſolvitur.

Problema. 2.

§. LXXIII. *Invenire numeros amicabiles hujus formæ* **a p q** & **a r s**,

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$\text{ſ}r$, *ubi* p, q, & r *ſint numeri primi*, *f ſive primus ſive compoſitus*, *qui perinde ac factor communis* a *ſit datus.*

Solutio.

Quærantur iterum ex cognito factore communi a valores b & c ut ſit $\frac{b}{c} = \frac{a}{2a - \text{ſ}a}$; & ſit numeri f ſumma diviſorum $\text{ſ}f = gh$. Cum igitur primo requiratur ut ſit $\text{ſ}p.\text{ſ}q = \text{ſ}f.\text{ſ}r$, erit $(p+1)(q+1) = gh(r+1)$. Ponatur $r+1 = xy$, $p+1 = hx$ & $q+1 = gy$, & neceſſe erit, ut ſint hi tres numeri primi, ſcilicet $p = hx - 1$; $q = gy - 1$ & $r = xy - 1$. Deinde opus eſt, ut ſit $\text{ſ}apq = ghxy\text{ſ}a = a(hx-1)(gy-1) + af(xy-1) = a((gh+f)xy - hx - gy + 1 - f)$; ſeu $2bghxy - cghxy = b(gh+f)xy - bhx - bgy + b(1-f)$ vel $(bf - bgh + cgh)xy - bhx - bgy = b(f-1)$.

Ponamus brevitatis gratia $bf - bgh + cgh = e$ erit $eexy - ebhx - ebgy = eb(f-1)$ ſive:

$$(ex - bg)(ey - bh) = bbgh + be(f-1)$$

Numerus ergo $bbgh + be(f-1)$ in duos ejuſmodi factores, qui ſint P & Q reſolvi debet, ut fiant

$$x = \frac{P + bg}{e} \quad \& \quad y = \frac{Q + bh}{e}$$ numeri integri, tum vero

$hx - 1$, $gy - 1$ & $xy - 1$ numeri primi. Quæ conditio, quoties impleri poterit, erunt numeri amicabiles $\begin{matrix} a(hx-1)(gy-1) \\ af(xy-1) \end{matrix}$

Notandum eſt, neque ullum horum numerorum primorum $hx - 1$, $gy - 1$, $xy - 1$, neque ullum factorem ipſius f diviſorem eſſe debere ipſius a: nec non f & $xy - 1$ eſſe debere numeros primos inter ſe.

Coroll. 1.

§. LXXIV. Si f ſit numerus primus, uti ſecunda forma numerorum amicabilium poſtulat; erit $f+1=gh$, & propterea $f=gh-1$. Hoc ergo caſu erit $e=cgh-b$ & $PQ=bbgh+be(gh-2)$ ſeu $PQ=bcgghh-2bcgh+2bb$. Unde quæri debent numeri x & y ſupra memoratis proprietatibus prædititi, ut ſit $x=\frac{P+bg}{e}$ & $y=\frac{Q+bh}{e}$,

Coroll. 2.

§. LXXV. His igitur formulis ita uti conveniet, ut pro a ſucceſſive alii atque alii valores ex iis, quos ſupra expoſui ſubſtituantur, atque pro ſingulis litteræ f varii numeri tam primi quam compoſiti ſubſtituantur, qui quidem ad numeros amicabiles inveniendos idonei videantur.

Caſus. 1.

§. LXXVI. Sit $a=4$, (ex valore enim $a=2$ nullos obtineri numeros amicabiles obſervavi) eritque $b=4$ & $c=1$. Tum poſitis numeris amicabilibus $4pq$ & $4fr$, ſit $f=gh$, & $e=4f-3gh$. Deinde per reſolutionem quærantur factores P & Q ut ſit:

$$PQ=16gh+4e(f-1)$$

Hincque eruantur numeri integri x & y, ut ſit

$$x=\frac{P+4g}{e} \quad \& \; y=\frac{Q+4h}{e}$$

& ex his deriventur valores litterarum $p=hx-1$, $q=gy-1$ & $r=xy-1$, qui ſi fuerint numeri primi, erunt $4pq$ & $4fr$ numeri amicabiles.

Exem-

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Exemplum. 1.

§. LXXVII. Sit $f = 3$, erit $ff = gh = 4$; hincque $e = 12 - 12 = 0$, unde patet ex hac hypotheſi nihil obtineri.

Exempl. 2.

§. LXXVIII. Sit $f = 5$, erit $ff = gh = 6$; $e = 20 - 18 = 2$, atque $PQ = 16.\,6 + 8.\,4 = 128$. Jam ex $gh = 6$ ponatur primo $g = 2$, & $h = 3$, fietque.

$$x = \frac{P+8}{2} \quad \& \quad y = \frac{Q+12}{2}$$

Quare ſequentes habebuntur reſolutiones:

$P =$	2	4	8	16	32	64
$Q =$	64	32	16	8	4	2
$x =$	5	6	8	12	20	36
$y =$	38	22	14	10	8	7
$p = 3x - 1 =$	19*	17	23	35*	59	107
$q = 2y - 1 =$	--	43	27*	19	15*	13
$r = xy - 1 =$	--	131	111*	119*	159	251

Hinc ergo prodeunt numeri amicabiles.

$\left\{\begin{matrix} 4.\,17.\,43 \\ 4.\,5.\,131 \end{matrix}\right)$ & $\left\{\begin{matrix} 4.\,13.\,107 \\ 4.\,5.\,251 \end{matrix}\right)$

Ponatur ſecundo $g = 1$, $h = 6$, fietque:

$$x = \frac{P+4}{2} \quad \& \quad y = \frac{Q+24}{2}$$

$P =$	2	4	8	16	32	64
$Q =$	64	32	16	8	4	2
$x =$	3	4	6	10	18	34
$y =$	44	28	20	16	14	13
$p = 6x - 1 =$	17*	23	35*	59	107	203*
$q = 1y - 1 =$	43	27*	19	15*	13	12*
$r = xy - 1 =$	131	111*	119*	159	251	441*

Iidem ergo prodeunt bini numeri amicabiles qui ante.

Sunt

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Sunt ergo hinc numeri amicabiles:

$$\begin{Bmatrix} 4\cdot 17\cdot 43 \\ 4\cdot 5\cdot 131 \end{Bmatrix} \quad \& \quad \begin{Bmatrix} 4\cdot 13\cdot 107 \\ 4\cdot 5\cdot 251 \end{Bmatrix}$$

Exempl. 3.

§. LXXIX. Sit $f = 7$, erit $ff = gh = 8$; $e = 28 + 24 = 4$. & $PQ = 16.8 + 16.6 = 224$.

Sit ergo primo $g = 2$, $h = 4$ erit

$$x = \frac{P+8}{4};\ y = \frac{Q+16}{4};\ p = 4x - 1;\ q = 2y - 1;$$
$$r = xy - 1.$$

P	4	8	28	56
Q	56	28	8	4
x	3	4	9	16
y	18	11	6	5
$4x-1$	11	15*	35*	63*
$2y-1$	35*	21*	11	9*
$xy-1$	53	42	53	79

Sit ſecundo $g = 1$, $h = 8$; erit $x = \frac{P+4}{4}$; $y = \frac{Q+32}{4}$ & $p = 8x - 1$; $q = y - 1$; $r = xy - 1$.

P	4	8	28	56
Q	56	28	8	4
x	2	3	8	15
y	22	15	10	9
$8x-1$	15*	23	63*	119*
$y-1$	21	14	9	8
$xy-1$	43	44*	79	134*

Hinc ergo nulli prodeunt numeri amicabiles.

Exem-

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Exemplum. 4.

§. LXXX. Sit $f = 11$, erit $gh = 12$, $e = 8$. $PQ = 16.$ $12 + 32.\ 10 = 512$, vel erit $(8x - 4g)(8y - 4h) = 512$, quæ æquatio deprimitur ad $(2x - g)(2y - h) = 32$, qua reſoluta erit $p = hx - 1$; $q = gy - 1$, & $r = xy - 1$. Sive autem hic ponatur $g = 1$, $h = 12$; ſive $g = 2$, $h = 6$; ſive $g = 3$, $h = 4$, nulli prodeunt numeri primi pro p, q & r.

Exemplum. 5.

§. LXXXI. Sit $f = 13$, erit $gh = 14$; $e = 10$; $PQ = 224 + 40.\ 12 = 704$, & $(10x - 4g)(10y - 4h) = 704$, quæ deprimitur ad $(5x - 2g)(5y - 2h) = 176$. Hinc autem nulli alii numeri amicabiles obtinentur niſi $\begin{Bmatrix} 4.\ 5.\ 251 \\ 4.\ 13.\ 107 \end{Bmatrix}$, qui jam ante (§. 78.) ſunt inventi. Simul vero jam patet, etiamſi pro f majores numeri primi ſtatuantur, nullos novos numeros amicabiles prodire, quoniam vel p vel q ſortietur valorem minorem, qui pro f aſſumi potuiſſet.

Exempl. 6.

§. LXXXII. Sit $f = 5.\ 13$, erit $gh = 6.\ 14 = 84$; $e = 8$; $PQ = 16.\ 84 + 32.\ 64 = 64.\ 53$ & $(8x - 4g)(8y - 4h) = 64.\ 53$ ſeu $(2x - g)(2y - h) = 4.\ 53$. Hincque invenietur in numeris primis: $p = 43$; $q = 2267$, & $r = 1187$; unde erunt numeri amicabiles $\begin{Bmatrix} 4.\ 43.\ 2267 \\ 4.\ 5.\ 13.\ 1187 \end{Bmatrix}$

Caſus. II.

§. LXXXIII. Sit $a = 2^3 = 8$, erit $b = 8$, $c = 1$, tum poſi-

poſitis numeris amicabilibus $8pq$ & $8fr$, & $ff = gh$ erit $e = 8f - 7gh$, atque

$$(ex - 8g)(ey - 8h) = 64gh + 8e(f - 1)$$

unde caſus ſunt dignoſcendi, quibus ſiunt numeri primi

$$p = hx - 1;\ q = gh - 1,\ \&\ r = xy - 1$$

Exemplum. 1.

§. LXXXIV. Sit $f = 11$ erit $gh = 12$, $e = 4$, atque
$(4x - 8g)(4y - 8h) = 64.\ 12 + 32.\ 10 = 64.\ 17$ ſeu
$(x - 2g)(y - 2h) = 4.\ 17 = 68$.
Hinc autem nulli numeri amicabiles reperiuntur.

Exempl. 2.

§. LXXXV. Sit $f = 13$, erit $gh = 14$; $e = 6$ atque
$(6x - 8g)(6y - 8h) = 64.14 + 48.\ 12 = 64.23$ ſeu
$(3x - 4g)(hy - 4h) = 16.\ 23$, verum etiam hæc hypotheſis eſt inutilis.

Exempl. 3.

§. LXXXVI. Sit $f = 17$, erit $gh = 18$; $e = 10$, atque
$(10x - 8g)(10y - 8h) = 64.17 + 40.\ 16 = 64.38$ ſeu
$(5x - 4g)(5y - 4h) = 32.\ 19$, hincque prodeunt numeri amicabiles. $\left\{\begin{matrix} 8.\ 23.\ 59 \\ 8.\ 17.\ 79 \end{matrix}\right)$

Exempl. 4.

§. LXXXVII. Magis foecunda eſt hypotheſis $f = 11.\ 23$ minor enim valor pro f in compoſitis ſubſtitui nequit, erit $gh = 12.\ 24$, $e = 8$ unde

(8x

$$(8x - 8g)(8y - 8h) = 64 \cdot 12 \cdot 24 + 64 \cdot 252$$

ſeu $(x - g)(y - h) = 540$. Hinc autem reperiuntur ſequentes numeri amicabiles.

$$\left\{ \begin{array}{l} 8.383.1907 \\ 8.11.23.2543 \end{array} \right) \quad \left\{ \begin{array}{l} 8.467.1151 \\ 8.11.23.1871 \end{array} \right) \quad \left\{ \begin{array}{l} 8.647.719 \\ 8.11.23.1619 \end{array} \right)$$

Hujusmodi numeris compoſitis pro f ponendis multi inſuper alii inveniuntur numeri amicabiles.

Scholion.

§. LXXXVIII. Ingens combinationum numerus, qui in hoc exemplo locum habet, anſam mihi præbuit ſolutionem in aliam formam redigendi commodiorem. Scilicet cum ſit; $e = bf - (b - c)gh$;

$$PQ = bbgh + bc(f - 1) = (ex - bg)(ey - bh)$$

ex formulis $x = \frac{P + bg}{e}$ & $y = \frac{Q + bh}{e}$ eliciuntur valores

$$p = \frac{hP + bgh}{e} - 1; \; q = \frac{gQ + bgh}{e} - 1; \; r = \frac{PQ + b(hP + gQ) + bbgh}{ee} - 1$$

Sit ergo ob $gh = ff$;

$$e = bf - (b - c)ff; \; L = bbff + bc(f - 1)$$

& $MN = Lff$ erit

$$p = \frac{M + bff}{e} - 1; \; q = \frac{N + bff}{e} - 1; \; r = \frac{L + b(M + N) + bbff}{ee} - 1$$

& nunc quæſtio eo reducitur, ut numerus Lff reſolvatur in duos factores M & N, quorum uterque quantitate bff auctus fiat diviſibilis

ſibilis per e, & ut quoti hinc reſultantes unitate minuti ſint numeri primi. Denique oportet ut ſit $r + 1 = \frac{(p+1)(q+1)}{ff}$ & r numerus primus. Hunc ergo calculum in nonnullis caſibus illuſtrabo.

Caſus. III.

§. LXXXIX. Sit $a = 2^4 = 16$; erit $b = 16$; $c = 1$; atque

$$e = 16f - 15ff;\; L = 256ff + 16e(f-1)$$
$$\& \; MN = Lff.$$

Numeri igitur primi eſſe debent:

$$p = \frac{M + 16ff}{e} - 1;\; q = \frac{N + 16ff}{e} - 1;\; r = \frac{L + 256ff + 16(M+N)}{ee} - 1,$$

quibus inventis erunt numeri amicabiles;

$16pq$ & $16fr$.

Exempl. I.

§. LXXXX. Sit $f = 17$ erit $ff = 18$; $e = 2$; $L = 1024.5$ & $MN = 1024.5.18 = 2^{11}.3^2.5$

$$p = \frac{M + 288}{2} - 1;\; q = \frac{N + 288}{2} - 1;\; r = \frac{512.19 + 16(M+N)}{4} - 1$$

ſeu ſit $M = 2m$; $N = 2n$ ut ſit

$mn = 2^9.3^2.5$ erit

$$p = m + 143;\; q = n + 143;\; \& \; r = 8(m+n) + 2431$$

qui tres numeri debent eſſe primi, ut numeri amicabiles ſint $16pq$ & $16.17.r$.

Hoc

Hoc autem ſuccedit duobus modis, primo ſi $m = 24$, $n = 960$ & ſecundo ſi $m = 96$ & $n = 240$; unde numeri amicabiles prodeunt:

$$\left\{\begin{matrix} 16.167.1103 \\ 16.17.10303 \end{matrix}\right) \quad \left\{\begin{matrix} 16.383.239 \\ 16.17.5119 \end{matrix}\right)$$

Exempl. 2.

§. LXXXXI. Sit $f = 19$, erit $\int f = 20$, $e = 4$; $L = 128.49$ & $MN = 512.5.49 = 2^9.5.7^2$. Ergo

$$p = \frac{M+320}{4} - 1; \quad q = \frac{N+320}{4} - 1; \quad r = \frac{128.59+16(N+N)}{16} - 1$$

ſeu ſit $M = 4m$ & $N = 4n$ ut ſit

$mn = 32.5.49 = 2^5.5.7^2$ erit

$p = m+79$; $q = n+79$ & $r = 4(m+n)+711$.

Hinc ſi $m = 70$, $n = 112$ prodeunt numeri amicabiles

$$\left\{\begin{matrix} 16.149.191 \\ 16.19.1439 \end{matrix}\right)$$

Exempl. 3.

§. LXXXXII. Sit $f = 23$ erit $\int f = 24$; $e = 8$, $L = 256.5.7$ & $MN = 2048.3.5.7 = 2^{11}.3.5.7$

$$p = \frac{M+16.24}{8} - 1; \quad q = \frac{N+16.24}{8} - 1;$$

$$r = \frac{256.59+16(M+N)}{64} - 1$$

ſeu ſit $M = 8m$; $N = 8n$; & $mn = 2^5.3.5.7$ erit

$p = m+47$; $q = n+47$; & $r = 2(m+n)+235$

Hinc tres caſus oriuntur; $\left\{\begin{matrix} m = 56; \\ n = 60; \end{matrix}\right.$ $\left\{\begin{matrix} m = 42; \\ n = 80; \end{matrix}\right.$ $\left\{\begin{matrix} m = 6 \\ n = 560 \end{matrix}\right.$

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& numeri amicabiles ſunt:

$$\begin{cases} 16.103.107 \\ 16.23.467 \end{cases} \quad \begin{cases} 16.89.127 \\ 16.23.479 \end{cases} \quad \begin{cases} 16.53.607 \\ 16.23.1367 \end{cases}$$

Exemplum. 4.

§. LXXXIII. Sit $f = 31$; erit $\int f = 32$; $L = 512.31$ & $MN = 2^{14}.31$; $p = \frac{M + 16.32}{16} - 1$; $q = \frac{N + 16.32}{16} - 1$;

$$r = \frac{16(M+N) + 512.47}{256} + 1$$

Sit ergo $M = 16m$; $N = 16n$ ut ſit $mn = 2^6.31$ erit

$$p = m + 31;\ q = n + 31;\ r = m + n + 93$$

Hinc autem nulli prodeunt numeri amicabiles.

Exempl. 5.

§. LXXXIV. Sit $f = 47$, $\int f = 48$ erit $e = 32$ & $L = 1024.5.7$ & $MN = 2^{14}.3.5.7$ unde

$$p = \frac{M + 16.48}{32} - 1;\ q = \frac{N + 16.48}{32} - 1;\ \&$$

$$r = \frac{16(M+N) + 1024.47}{1024} - 1$$

Sit $M = 32m$ & $N = 32n$; ut ſit $mn = 2^4.3.5.7$ erit $p = m + 23$; $q = n + 23$; $r = \frac{1}{2}(m+n) + 46$. Ergo $m+n$ debet eſſe numerus impariter par, ut $\frac{1}{2}(m+n)$ fiat impar, quod evenit ſi vel m vel n ſit impariter par. Sit

$m = 30$; $n = 56$ erunt N. Amic. $\begin{cases} 16.53.79 \\ 16.47.89 \end{cases}$

Exem-

Exempl. 6.

§. XCIV. Sit $f = 17.137$, erit $ff = 18.138 = 4.27.23 = 2484$; $e = 4$; $L = 256.2484 + 64.2328 = 512.3.7.73$ &
$MN = 2048.81.7.23.73$

$$p = \frac{M + 16.2484}{4} - 1; \quad q = \frac{N + 16.2484}{4} - 1;$$

$$r = \frac{512.2775 + 16(M+N)}{16} - 1$$

Sit $M = 4m$; $N = 4n$ erit $mn = 128.81.7.23.73$ &
$p = m + 9935$; $q = n + 9935$ & $r = 4(m+n) + 88799$

Sed hic ſemper prodit valor ipſius r major quam 100000, ita ut difficile ſit diſcernere, utrum ſit primus nec ne.

Exempl. 7.

§. XCV. Sit $f = 17.151$ erit $ff = 18.152 = 16.9.19 = 2736$, $e = 32$; & $L = 1024.1967 = 1084.7.281$. atque
$MN = 2^{14}.9.7.19.281$.
Sit $M = 32m$; $N = 32n$ erit $mn = 16.9.7.19.281$ &
$p = m + 1367$; $q = n + 1367$; $r = \frac{1}{2}(m+n) + 2650$
Sit $m = 2\mu$, $n = 8\nu$, erit $\mu\nu = 9.7.19.281$ &.
$p = 2\mu + 1367$; $q = 8\nu + 1367$; $r = \mu + 4\nu + 2650$.

Hinc primum patet neque μ neque ν eſſe poſſe numerum formæ $3\alpha + 2$; tum μ non poſſe deſinere in 9 nec ν in 1; quibus obſervatis ſequentes tantum reſolutiones locum habent.

	*		*		*	*	
μ	3.281	7.19	21.281	21	63.281	3	1
ν	21.19	9.281	57	57.281	19	399.281	1197.281

quo-

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quorum ii, qui aſteriſcis ſunt notati, excluduntur ideo, ne p, q, vel r fiat per 7 diviſibile. Quarta reſolutio dabit hos numeros amicabiles $\left\{ \begin{array}{l} 16.\ 1409.\ 129503 \\ 16.\ 17.\ 151.\ 66739 \end{array} \right)$, ſi modo hic numerus 129503 eſt primus.

Exemplum. 8.

§. XCVI. Sit $f = 17.\ 167$, erit $ff = 18.\ 168 = 16.\ 27.\ 7 = 3024$, $e = 64$: $L = 2048.\ 1797 = 2048.\ 3.\ 599$ &

$MN = 2^{15}.\ 3^4.\ 7.\ 599$

ſit $M = 64\,m$; $N = 64\,n$, erit $mn = 2^3.\ 3^4.\ 7.\ 599$ &

$$p = m + 755;\ q = n + 755;\ r = \tfrac{1}{4}(m+n) + \frac{2173}{2}$$

ſit $m = 2\mu$; $n = 4\nu$, erit $\mu\nu = 3^4.\ 7.\ 599$ &

$$p = 2\mu + 755;\ q = 4\nu + 755;\ r = \nu + \frac{\mu+1}{2} + 1086$$

Ubi patet eſſe oportere $\mu = 4\alpha - 1$, ne r fiat numerus par: nec $\mu = 3\alpha + 2$, nec $\nu = 3\alpha + 1$. Hinc prodeunt numeri amicabiles $\left\{ \begin{array}{l} 16.\ 809.\ 51071 \\ 16.\ 17.\ 167.\ 13679 \end{array} \right)$

Caſus. IV.

§. XCVII. Sit vel $a = 3^3.\ 5$ vel $a = 3^2.\ 7.\ 13$, ut ſit $b = 9$, $c = 2$ erit $e = 9f - 7ff$: $L = 81.\ ff + 9e(f-1)$ & $MN = Lff$ erit

$$p = \frac{M + 9ff}{e} - 1;\ q = \frac{N + 9ff}{e} - 1;$$

$$r = \frac{9(M+N) + L + 81ff}{ee} - 1$$

qui

qui numeri p, q, r ſi fuerint primi erunt numeri amicabiles.

$$\left\{ \begin{matrix} apq \\ afr \end{matrix} \right]$$

Exemplum.

§. XCVIII. Sit $f = 7$; $ſf = 8$, erit $e = 7$, $L = 2.27.19$; $MN = 16.27.19$, erit $p = \frac{M+72}{7} - 1$; $q = \frac{N+72}{7} - 1$;

$$r = \frac{9(M+N)+2.27.31}{49} - 1$$

Unde poſito $M = 54$, $N = 152$ oriuntur numeri amicabiles

$$\begin{matrix} a.\ 17.\ 31 \\ a.\ 7.\ \ 71 \end{matrix} \quad \text{ſeu} \quad \left\{ \begin{matrix} 3^3.5.\ 17.\ 31 \\ 3^3.5.\ 7.\ \ 71 \end{matrix} \right)$$

Problema. 4.

§. XCIX. *Invenire numeros amicabiles hujus formæ:* agpq, & ahr, *ubi* p, q, r *ſint numeri primi, at* g & h *ſive primi ſive compoſiti dati, una cum factore communi* a.

Solutio.

Ex factore communi a quæratur in minimis terminis fractis $\frac{b}{c} = \frac{a}{2a - ſa}$; deinde ſit $\frac{ſg}{ſh} = \frac{m}{n}$; & ex prima proprietate numerorum amicabilium erit $(p+1)(q+1)ſg = (r+1)ſh$ ſeu $r+1 = \frac{m}{n}(p+1)(q+1)$. Altera vero proprietas præbet: $(r+1)ſa.ſh = a(gpq+hr)$, vel ob $\frac{ſa}{a} = \frac{2b-c}{b}$ erit $(r+1)(2b-c)ſh = b(gpq+hr)$ & pro r ſubſtituto valore

$m(2b-c)(p+1)(q+1)ſh = b(ngpq + mh(p+1)(q+1) - nh).$
Sit brevitatis gratia $p+1 = x$; $q+1 = y$ erit:

$m(2b-c)xyſh = b(mhxy + ngxy - ngx - ngy + ag - nh)$ vel

$$\left.\begin{array}{l} mbh \\ nbg \\ -2mbſh \\ +mcſh \end{array}\right\} xy - nbgx - nbgy = nb(h-g)$$

Ponatur brevitatis gratia $e = b(mh+ng) - (2b-c)mſh$
eritque $eexy - nbgex - nbgey + nnbbgg = nnbbgg + nb(h-g)e$
ſeu $(ex - nbg)(ey - nbg) = nnbbgg + nb(h-g)e$

Ponatur ergo $nnbbgg + nb(h-g)e = MN$ fietque

$$x = \frac{M+nbg}{e} \;\&\; y = \frac{N+nbg}{e} \quad \text{ſeu}$$

$$p = \frac{M+nbg}{e} - 1, \;\&\; q = \frac{N+nbg}{e} - 1; \;\&\; r = \frac{m}{n}xy - 1$$

Qui tres numeri p, q, & r ſi fuerint primi, erunt numeri amicabiles $agpq$ & ahr, dummodo utriusque factores ſint primi inter ſe.

Coroll.

§. C. Si ſint g & h numeri primi: erit $\frac{m}{n} = \frac{g+1}{h+1}$; ſit ergo $g = km - 1$ & $h = kn - 1$; erit $ſh = kn$, unde fiet
$e = b(2kmn - m - n) - (2b-c)kmn = ckmn - b(m+n)$
$MN = nb(nb(km-1)^2 + k(n-m)e) = (ex - bn(km-1))$ $(ey - bn(km-1))$ & $p = x - 1$; $q = y - 1$ atque $r = \frac{m}{n}$ $xy - 1$;

Ca-

Caſus. I.

§. CI. Sit $m = 1$; $n = 3$; ergo $g = k - 1$; $h = 3k - 1$; eritque $e = 3ck - 4b$; & $MN = 3b(3b(k-1)^2 + 2ke)$ ideoque

$$x = \frac{M + 3b(k-1)}{e}; \quad y = \frac{N + 3b(k-1)}{e}$$

ac denique $p = x - 1$; $q = y - 1$; & $r = \frac{1}{3}xy - 1$.

Exempl. I.

§. CII. Sit $a = 4$; $b = 4$; $c = 1$; erit $e = 3k - 16$; & $MN = 12(12(k-1)^2 + 2ke)$ & $x = \frac{M + 12(k-1)}{e}$ & $y = \frac{N + 12(k-1)}{e}$. Hic poni poteſt

I. $k = 6$, fietque $g = 5$, $h = 17$, & $e = 2$, ſed hinc nihil efficitur

II. $k = 8$, fietque $g = 7$, $h = 23$; & $e = 8$, $MN = 12(12.49 + 128)$ ſeu $MN = 16.3.179 = (8x - 84)(8y - 84)$ ideoque $3.179 = (2x - 21)(2y - 21$ unde nihil pariter ſequitur:

Exempl. 2.

§. CIII. Sit $a = 8$; $b = 8$, $c = 1$; erit $e = 3k - 32$; $MN = 24(24(k-1)^2 + 2ke)$ ſeu $MN = 48(15kk - 56k + 12) = (ex - 24(k-1))(ex - 24(k-1))$

Verum ne hinc quoque quicquam concludere licet.

Caſus. II.

§. CIV. Sit $m = 3$; $n = 1$, erit $e = 3ck - 4b$; & $g = 3k - 1$; $h = k - 1$

$MN = b(b(3k-1)^2 - 2ke) = (ex - b(3k-1))(ey - b(3k-1))$

atque $p = x - 1$; $q = y - 1$, & $r = 3xy - 1$.

 Exem-

Exemplum. 1.

§. CV. Sit $a = 10$, $b = 5$, $c = 1$, erit $e = 3k - 20$, &
$$5(5(3k-1)^2 - 2ke) = (ex - 5(3k-1))(ey - 5(3k-1)).$$
Si hic ponatur $k = 8$, fiet $5 . 29 . 89 = (4x - 115)(4y - 115)$.

Unde prodit $x = 30$, $y = 674$, $3xy = 60660$, & numeri amicabiles erunt;

$$\begin{Bmatrix} 10 . 23 . 29 . 673 \\ 10 . 7 . 60659 \end{Bmatrix}$$

Exempl. 2.

§. CVI. Sit $a = 3^3 . 5$, $b = 9$, $c = 2$, erit $e = 6k - 36$; &
$$(9(3k-1)^2 - 3ke) = (\tfrac{1}{3}ex - 3(3k-1))(\tfrac{1}{3}ey - 3(3k-1))$$
Jam fiat $k = 8$, erit $e = 12$; & $3 . 1523 = (4x - 69)(4y - 69)$ hincque oritur $x = 18$, $y = 398$, $3xy = 21492$ eruntque numeri primi; $g = 23$; $h = 7$; $p = 17$; $q = 397$; $r = 21491$ & numeri amicabiles:

$$\begin{Bmatrix} 3^3 . 5 . 23 . 17 . 397 \\ 3^3 . 5 . 7 . 21491 \end{Bmatrix}$$

Scholion.

§. CVII. Ex his exemplis uſus hujus problematis in inveniendis numeris amicabilibus ſatis luculenter perſpicitur; ſed ob ipſam nimiam fingendi libertatem non parum moleſtum eſt ſecundum præcepta hic tradita omnes caſus percurrere. Cum igitur ſufficiat hanc methodum tradidiſſe, ejusque uſum monſtraſſe, ei prolixius non immoror, ſed ad ultimam methodum, cujus ope numeros amicabiles eruere liceat, qua quidem ſum uſus, exponendam progredior. Nititur ea autem ſingularibus proprietatibus, quibus numeri ratione ſummæ diviſorum gaudent, quas oblata occaſione explicabo, ne plurium lemmatum præmiſſio tædium creet.

Iis

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Iis autem expoſitis non difficile erit plura alia problemata ad hoc genus pertinentia reſolvere.

Problema 5.

§. CVIII. *Invenire numeros amicabiles hujus formæ:* zap & zbq, *ubi factores* a & b *ſint dati,* p & q *numeri primi, & factor communis* z *inveſtigari debeat.*

Solutio.

Sit $\int a : \int b = m : n$: & cum eſſe debeat $\int a . (p + 1) = \int b . (q + 1)$, erit $m(p + 1) = n(q + 1)$ Ponatur $p + 1 = nx$ & $q + 1 = mx$, eruntque numeri amicabiles $za(nx - 1)$ & $zb(mx - 1)$. Ubi quidem requiritur ut $mx - 1$ & $nx - 1$ ſint numeri primi. Cum jam utriusque numeri eadem ſit ſumma diviſorum $= nx\int a . \int z = mx\int b . \int z$ oportet ut ea ſit æqualis ſummæ numerorum $z((na + mb)x - a - b)$. Unde obtinetur iſta æquatio $\frac{z}{\int z} = \frac{nx\int a}{(na + mb)x - a - b}$. Quo jam ex hac æquatione valor ipſius z elici queat, fractio $\frac{nx\int a}{(na + mb)x - a - b}$ ad minimos terminos reducatur, quæ ſit $= \frac{r}{s}$, ita ut habeatur $\frac{z}{\int z} = \frac{r}{s}$: hincque ſequentia ſunt notanda. Primo eſſe z vel ipſi r æquale, vel ejus multiplo cuipiam puta kr. Priori caſu ſi $z = r$ erit $\int z = s$ ac propterea $s = \int r$. Poſteriori caſu, ſi $z = kr$ erit $\int z = ks = \int kr$: Verum quicquid ſit k, erit $\frac{\int kr}{\int r} > k$, nam $\int kr$ continet omnes diviſores ipſius r ſingulos per k multiplicatos, & inſuper eos diviſores ipſius r, qui non ſunt per k diviſibiles; eritque ergo $\int kr > k\int r$.

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$k\int r$. Cum igitur ſit $\int z > k\int r$, erit quoque $ks > k\int r$ ſeu $s > \int r$. Quare ſi in fractione $\frac{r}{s}$ fuerit $s = \int r$, erit $z = r$; ſin autem ſit $s > \int r$ erit z æquale multiplo cuipiam ipſius r. Unde patet ſi ſit $s < \int r$ æquationem $\frac{z}{\int z} = \frac{r}{s}$ eſſe impoſſibilem, neque hoc caſu numeros amicabiles inveniri poſſe. Deinde cum ſit $\frac{\int z}{z} = \frac{na+mb}{n\int a} - \frac{a-b}{nx\int a} = \frac{a}{\int a} + \frac{b}{\int b} - \frac{a-b}{nx\int a}$; ob $\frac{a}{\int a} < 1$ & $\frac{b}{\int b} < 1$ erit $\frac{\int z}{z} < 2 - \frac{a-b}{nx\int a}$, ideoque multo magis $\frac{z}{\int z} > \frac{1}{2}$ ita ut z ſit ſemper numerus deficiens. Hincque patet æquationem $\frac{z}{\int z} = \frac{r}{s}$ ſemper ita fore comparatam, ut ſit $\frac{r}{s} > \frac{1}{2}$ ſeu $s < 2r$. Unde ſi ſit $\int r = s$, erit $\int r < 2r$, &, ſi $s > \int r$ erit multo magis $\int r < 2r$. Utroque igitur caſu r erit numerus deficiens. Quocirca ſi x tanquam numerus incognitus ſpectetur, propoſita æquatione $\frac{z}{\int z} = \frac{nx\int a}{(na+mb)x - a - b}$, valorem ipſius x ita determinari oportet; ut reducta fractione $\frac{nx\int a}{(na+mb)x - a - b}$, ad minimos terminos $\frac{r}{s}$, fiat r numerus deficiens, & ut ſit vel $s = \int r$ vel $s > \int r$. Quibus conditionibus animadverſis, tam r quam s in ſuos factores ſimplices primos reſolvatur, ut prodeat hujusmodi æquatio $\frac{z}{\int z} = \frac{A^{\alpha} B^{\beta} C^{\gamma}}{E^{\epsilon} F^{\zeta} G^{\eta}}$, tunc autem ſucceſſive vel

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vel A^{α} vel altior poteſtas ipſius A ponatur factor ipſius z, ſeu ponatur $z = P.\, A^{\alpha+\nu}$ erit $\int z = \int A^{\alpha+\nu} \int P$ & $\frac{z}{\int z} = \frac{P\, A^{\alpha+\nu}}{\int A^{\alpha+\nu} \,.\, \int P}$,

ideoque $\frac{P}{\int P} = \frac{B^{\zeta}\, C^{\gamma} \int A^{\alpha+\nu}}{A^{\nu}\, E^{\iota}\, F^{\xi}\, G^{\eta}}$. Similique modo ponatur ulterius $P = B^{\zeta+\mu}\, Q$, & hoc pacto procedatur, donec tandem perveniatur ad æquationem hujus formæ $\frac{Z}{\int Z} = \frac{u}{\int u}$, ex qua habeatur $Z = u$. Sæpe quidem hæc operatio ſucceſſu optato caret, ſed pro quovis caſu oblato facilius erit operationem hanc per exempla docere, quam per præcepta.

Exemplum. I.

§. CIX. Sit $a = 3$; $b = 1$ erit $\int a = 4$; $\int b = 1$; & $m = 4$; $n = 1$ ac numeri amicabiles erunt: $3(x-1)z$ & $(4x-1)z$, ſi ſint $x-1$ & $3x-1$ numeri primi & $\frac{z}{\int z} = \frac{4x}{7x-4}$. Hic autem primo patet, ſi 4 ex numeratore non tollatur, fore $7x-4 < \int 4x$ ob $\int 4x = 7\int x$. Ergo neceſſe eſt ut ſit $7x-4$ numerus par. Ponatur $x = 4p$; erit $\frac{z}{\int z} = \frac{4p}{7p-1}$. Nunc fiat $7p-1$ numerus par, ponendo $p = 2q+1$, erit $\frac{z}{\int z} = \frac{2(2q+1)}{7q+3}$; & $x = 8q+4$; atque $x-1 = 8q+3$; $4x-1 = 32q+15$.

Unde

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Unde q nequit eſſe multiplum ternarii, ne $x - 1$ fiat per 3 diviſibile. Erit ergo vel $q = 3r + 1$, vel $q = 3r - 1$, priori caſu fit $2q + 1 = 6r + 3$, ac z deberet eſſe diviſibile per 3, quod pariter fieri nequit, quia in altero numero quæſito $3(x - 1)z$ jam ineſt factor 3. Sit igitur $q = 3r - 1$, erit $\frac{z}{\int z} = \frac{2(6r - 1)}{21r - 4}$ atque $x = 24r - 4$; $x - 1 = 24r - 5$; & $4x - 1 = 96r - 17$. Cum autem z factorem 4 habere nequeat, niſi binarius ex numeratore $2(6r - 1)$ tollatur, z erit diviſibile per 2, & poſito $z = 2y$ fierit $\frac{2y}{3\int y} = \frac{2(6r - 1)}{21r - 4}$, & $\frac{y}{\int y} = \frac{3(6r - 1)}{21r - 4}$: ideoque evaderet y & propterea quoque z per 3 divibile quod fieri nequit. Hanc obrem iſte binarius ex numeratore tolli debet, ponendo $r = 2s$; ut ſit $x - 1 = 48s - 5$; $4x - 1 = 192s - 17$ eritque $\frac{z}{\int z} = \frac{12s - 1}{21s - 2}$. Jam ſi s ſit numerus impar ob z numerum imparem, fiet quoque $\int z = k(21s - 2)$ numerus impar, ex quo ſequitur, numerum z fore quadratum: ſin autem s ſit numerus par, factor communis z non erit quadratus. Evolvantur ergo ii ipſius s valores, qui efficiunt $x - 1 = 48s - 5$ & $4x - 1 = 192s - 17$ numeros primos; & diſpiciatur utrum æquationi $\frac{z}{\int z} = \frac{12s - 1}{21s - 2}$ ſatisfieri queat. Sit $s = 7$; erit $x - 1 = 331$, $4x - 1 = 1327$ & $\frac{z}{\int z} = \frac{83}{145}$. Jam cum z debeat eſſe quadratum, ponatur $z = 83^2 A$, erit $\int z = 367 \cdot 19 \int A$ & $\frac{A}{\int A} = \frac{367 \cdot 19}{5 \cdot 29 \cdot 83}$. Nunc autem ipſius A factor ſtatui nequit 19^2, ob $\int 19^2 =$

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§. 127, prodiret enim 3 factor ipſius A, altioribus vero poteſtatibus ſumendis, mox devenitur ad numeros tam grandes, ut facile pateat opus ſuccedere non poſſe.

Si $s = 12$; erit $x - 1 = 571$; $4x - 1 = 2287$ & $\frac{z}{\int z} = \frac{11.13}{2.\ 125}$ quæ neque 11^2 neque 13 pro factoribus ipſius z aſſumendo reſolvi poteſt.

Neque vero etiam ex majoribus valoribus pro s mihi quicquam præſtare licuit.

Exempl. 2.

§. CX. Sit $a = 5$, $b = 1$; erit $\int a = 6$; $\int b = 1$; $m = 6$, $n = 1$ & numeri amicabiles erunt $5(x - 1)z$ & $(6x - 1)z$, habebiturque $\frac{z}{\int z} = \frac{6x}{11x - 6}$. Quæ æquatio ut ſit poſſibilis ex numeratore $6x$ vel binarium, vel ternarium tollere oportet, quia alioquin numerator maneret numerus redundans. Habebimus ergo duos caſus evolvendos.

1. Tollatur ex numeratore ternarius ponendo $x = 3p$, erit $\frac{z}{\int z} = \frac{6p}{11p - 2}$, nunc vero porro ponatur $p = 3q + 1$, eritque $\frac{z}{\int z} = \frac{2(3q + 1)}{11q + 3}$ & ob $x = 9q + 3$ numeri primi eſſe debeat $x - 1 = 9q + 2$, & $6x - 1 = 54q + 17$, ubi patet q eſſe debere numerum imparem. Sit ergo $q = 2r - 1$: erit $x - 1 = 18r - 7$; $6x - 1 = 108r - 37$

$$\& \quad \frac{z}{\int z} = \frac{2(6r - 2)}{22r - 8} = \frac{2(3r - 1)}{11r - 4}.$$

Evolvantur jam caſus quibus $18r - 7$ & $108r - 37$ ſiunt numeri primi: qui ſunt:

1). $r=1$; erit $x-1=11$; $6x-1=71$; & $\frac{z}{\int z}=\frac{2.2.}{4}=\frac{4}{7}$

Cum igitur hic ſit $7=\int 4$, erit $z=4$, & numeri amicabiles erunt $\left\{\begin{array}{l}4.5.11\\4.71\end{array}\right)$, quos quidem jam invenimus:

2). $r=2$; erit $x-1=29$, $6x-1=179$ & $\frac{z}{\int z}=\frac{2.5.}{2.9.}=\frac{5}{9}$. At z factorem 5 habere nequit.

3). $r=5$; erit $x-1=83$, $6x-1=503$ & $\frac{z}{\int a}=\frac{4.7}{3.17}$; at hic $3.17<\int 4.7$.

4). $r=8$; erit $x-1=137$; $6x-1=827$ & $\frac{z}{\int z}=\frac{23}{2.3.7}$;

Ponatur $z=4.23P$, erit $\int z=24\int P$ & $\frac{P}{\int P}=\frac{24}{23}.\ \frac{z}{\int z}=\frac{4}{7}$; unde $P=4$, & $z=4.23$: quam operationem ita breviter repræſento $\frac{z}{\int z}=\frac{23}{2.3.7.}\left|\frac{23}{24}\right|\frac{4}{7}\left|\frac{4}{7}\right|$, unde ſit $z=4.23$ & numeri amicabiles erunt.

$$\left\{\begin{array}{l}4.23.5.137\\4.23.827\end{array}\right)$$

Reliqui valores, quousque quidem examinavi nullos dant numeros amicabiles.

II. Tollatur ex numeratore binarius, ponendo $x=2p$ erit $\frac{z}{\int z}=\frac{6p}{11p-3}$; Nunc ſit $p=2q+1$; erit $\frac{z}{\int z}=\frac{3(2q+1)}{11q+4}$: & numeri primi eſſe debebunt (ob $x=4q+2$); $x-1=4q+1$;

6x-

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$6x-1=24q+11$: quare eſſe nequit $q=3\alpha-1$. Deinde cum z non eſſe debeat diviſibile per 5, neque $2q+1$, neque $4q+1$, neque $24q+11$ per 5 debet eſſe diviſ: unde excluduntur caſus $q=5\alpha+2$; $q=5\alpha+1$. Rejectis ergo his aliisque valoribus inutilibus ipſius q, qui pro $x-1$ & $6x-1$ non præbent numeros primos, calculus ita ſi habebit

q	$x-1$	$6x-1$	$\frac{z}{\int z}=\frac{3(2q+1)}{11q+4}$
3	13	83	$\frac{3\cdot 7}{37}$ nihil dat
4	17	107	$\frac{3\cdot 9}{48}=\frac{9}{16}\left\vert\frac{9}{13}\right\vert\frac{13}{16}\left\vert\frac{13}{14}\right\vert\frac{7}{8}\left\vert\frac{7}{8}\right\vert$; $z=9\cdot 7\cdot 13$
			vel $\frac{9}{16}\left\vert\frac{27}{40}\right\vert\frac{5}{6}\left\vert\frac{5}{6}\right\vert$ ergo $z=27\cdot 5$. Hic autem valor ob $a=5$ eſt inutilis; erunt ergo numeri amicabiles $\begin{cases} 9\cdot 7\cdot 13\cdot 5\cdot 17 \\ 9\cdot 7\cdot 13\cdot 107 \end{cases}$
9	37	227	$\frac{3\cdot 19}{103}$ nihil dat
10	41	251	$\frac{3\cdot 21}{114}=\frac{3\cdot 7}{2\cdot 19}\left\vert\frac{7^2}{3\cdot 19}\right\vert\frac{3^2}{2\cdot 7}\left\vert\frac{3^2}{13}\right\vert\frac{13}{14}\left\vert\frac{13}{14}\right\vert$
			Ergo $z=3^2\cdot 7^2\cdot 13$ & numeri amicabiles erunt $\begin{cases} 3^2\cdot 7^2\cdot 13\cdot 5\cdot 41 \\ 3^2\cdot 7^2\cdot 13\cdot 251 \end{cases}$

18	73	443	$\frac{3\cdot 37}{202} = \frac{3\cdot 37}{2\cdot 101}$ nihil dat.
24	97	587	$\frac{3\cdot 49}{268} = \frac{3\cdot 49}{4\cdot 67}$ nihil dat.
28	113	683	$\frac{3\cdot 57}{312} = \frac{9\cdot 19}{8\cdot 39} = \frac{3\cdot 19}{8\cdot 13}$ nihil dat
34	137	827	$\frac{3\cdot 69}{378} = \frac{23}{2\cdot 21} = \frac{23}{2\cdot 3\cdot 7}\left\|\frac{23}{24}\right\|\frac{4}{7}\left\|\frac{4}{7}\right\|$; $z = 4\cdot 23$; ut ante
39	157	947	$\frac{3\cdot 79}{433}$ nihil dat.
45	181	1091	$\frac{3\cdot 91}{499} = \frac{3\cdot 7\cdot 13}{499}$
48	193	1163	$\frac{3\cdot 97}{532} = \frac{3\cdot 97}{4\cdot 7\cdot 19} = \frac{3\cdot 97}{4\cdot 133}\left\|\frac{97}{2\cdot 7}\right\|\frac{3\cdot 7}{2\cdot 19}\left\|\frac{7^2}{3\cdot 19}\right\|\frac{3^2}{2\cdot 7}\left\|\frac{3^2}{13}\right\|\frac{13}{14}$ Ergo $z = 3^2\cdot 7^2\cdot 13\cdot 97$. & numeri amicabiles ſunt $\left\{\begin{array}{l} 3^2\cdot 7^2\cdot 13\cdot 97\cdot 5\cdot 193 \\ 3^2\cdot 7^2\cdot 13\cdot 97\cdot 1163 \end{array}\right)$
49	197	1187	$\frac{3\cdot 99}{543} = \frac{9\cdot 11}{181}$
60	241	1451	$\frac{3\cdot 121}{664} = \frac{3\cdot 11^2}{8\cdot 83}$

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69	277	1667	$\frac{3\cdot 139}{793}$
79	317	1907	$\frac{3\cdot 159}{873} = \frac{53}{97}$
84	337	2027	$\frac{3\cdot 169}{928} = \frac{3\cdot 169}{8\cdot 116} = \frac{3\cdot 169}{32\cdot 29}$
93	373	2243	$\frac{3\cdot 187}{1027} = \frac{3\cdot 11\cdot 17}{13\cdot 79}$
100	401	2411	$\frac{3\cdot 201}{1104} = \frac{3\cdot 67}{368} = \frac{3\cdot 67}{16\cdot 23}$
244	977	5867	$\frac{3\cdot 489}{2688} = \frac{3\cdot 163}{128\cdot 7}$ $\left|\frac{163}{4\cdot 41}\right|$ $\frac{3\cdot 41}{32\cdot 7}$ $\left|\frac{41}{2\cdot 3\cdot 7}\right|$ $\frac{3^2}{16}$
			$\left|\frac{3^2}{13}\right|$ $\frac{13}{16}$ $\left|\frac{13}{14}\right|$ $\frac{7}{8}$

Ergo $z = 3^2\cdot 7\cdot 13\cdot 41\cdot 163$ & numeri amicabiles erunt

$$\begin{cases} 3^2\cdot 7\cdot 13\cdot 41\cdot 163\cdot 5\cdot 977 \\ 3^2\cdot 7\cdot 13\cdot 41\cdot 163\cdot 5867 \end{cases}$$

Hinc ergo bini prodierunt novi numeri amicabiles.

Exempl. 3.

§. CXI. Sit $a = 7$; $b = 1$; erit $\int a = 8$, $\int b = 1$; $m = 8$,

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$p = 1$ & numeri amicabiles: $7(x-1)z$ & $(8x-1)z$, exiſtente $\frac{z}{\int z} = \frac{8x}{15x-8}$. Ac primo quidem x debet eſſe numerus par: ponatur ergo $x = 2p$; erit $x - 1 = 2p - 1$; $8x - 1 = 16p - 1$ & $\frac{z}{\int z} = \frac{8p}{15p-4}$: quæ æquatio eſt impoſſibilis, niſi poteſtas binarii in numeratore deprimatur, quia $15p - 4 < \int 8p$. Ergo fiat $p = 4q$, ut ſit $x = 8q$; $x - 1 = 8q - 1$; $8x - 1 = 64q - 1$ & $\frac{z}{\int z} = \frac{8q}{15q-1}$. Nunc ſit $q = 2r + 1$; erit $\frac{z}{\int z} = \frac{4(2r+1)}{15r+7}$ & $x - 1 = 16r + 7$; $8x - 1 = 128r + 63$, quorum numerorum ut neuter ſit per 3 diviſibilis, neque erit $r = 3\alpha - 1$, neque $r = 3\alpha$. Sit ergo $r = 3s + 1$; erit $\frac{z}{\int z} = \frac{4(6s+3)}{45s+22}$ ſeu $\frac{z}{\int z} = \frac{4 \cdot 3(2s+1)}{45s+22}$ & $x - 1 = 48s + 23$; $8x - 1 = 384s + 191$

Nunc vel ternarius vel quaternarius ex numeratore tolli debet. At ternarius tolli nequit, quia denominator nunquam per 3 eſt diviſibilis; tollatur ergo quaternarius, ad quod pono $s = 2t$ eritque $\frac{z}{\int z} = \frac{2 \cdot 3(4t+1)}{45t+11}$; nunc ſit $t = 2u - 1$; erit $\frac{z}{\int z} = \frac{3(8u-3)}{45u-17}$: at eſt $s = 4u - 2$, ideoque numeri primi eſſe debent $x - 1 = 192u - 73$; $8x - 1 = 1536u - 577$

1)u

	$x-1$	$8x-1$	$\frac{z}{\int z}$
1) $u=5$	887	7103	$\frac{3.37}{208}=\frac{3.37}{16.13}$ $\left\lvert\frac{37}{2.19}\right\rvert$ $\frac{3.19}{8.13}$ $\left\lvert\frac{19}{4.5}\right\rvert$ $\frac{3.5}{2.13}$ $\left\lvert\frac{5}{2.3}\right\rvert$ $\frac{3^2}{13}$ Ergo $z=3^2.5.19.37$ & numeri amicabiles erunt: $\left\{\begin{matrix}3^2.5.19.37.7.887\\3^2.5.19.37.7103\end{matrix}\right)$;
$u=11$	2039	16319	$\frac{3.5.17}{4.107}$
$u=13$	2423	19391	$\frac{3.101}{8.71}$
$u=26$	4919	39359	$\frac{3.205}{1153}$

Exemplum. 4.

§. CXII. Sit $a=11$, $b=1$, erit $\int a=m=12$; $\int b=n=1$; numeri quæſiti $11(x-1)z$ & $(12x-1)z$; atque $\frac{z}{\int z}=\frac{12x}{23x-12}$. Hic ex numeratore vel 3 vel 4 tolli debet:

I. Tollatur 3; ponatur $x=3p$; erit $\frac{z}{\int z}=\frac{12p}{23p-4}$; & $p=$

$p = 3q - 1$, erit $\frac{z}{\int z} = \frac{4(3q-1)}{23q-9}$; & ob $x = 9q - 3$, q debet eſſe impar. Sit $q = 2r + 1$, ut ſit $x = 18r + 6$, erit $\frac{z}{\int z} = \frac{4(6r+2)}{46r+14} = \frac{4(3r+1)}{23r+7}$, & $x - 1 = 18r + 5$; $12x - 1 = 216r + 71$.

r	$x-1$	$12x-1$	$\frac{z}{\int z}$
0	5	71	$\frac{4}{7}$; $z = 4$; num. amic. $\begin{cases} 4 \cdot 11 \cdot 5 \\ 4 \cdot 71 \end{cases}$
2	41	503	$\frac{4 \cdot 7}{53}$
3	59	719	$\frac{4 \cdot 10}{76} = \frac{2 \cdot 5}{19}$ imp.
6	113	1367	$\frac{4 \cdot 19}{145} = \frac{4 \cdot 10}{5 \cdot 29}$ imp.
7	131	1583	$\frac{4 \cdot 22}{168} = \frac{11}{21} = \frac{11}{3 \cdot 7} \left\| \frac{11}{12} \right\| \frac{4}{7}$ ſed ob factorem 11 hic valor z non valet.

II. Tollatur factor 4, ac ponatur $x = 4p$; ut fiat $\frac{z}{\int z} = \frac{12p}{23p - 3}$. Jam ſit $p = 4q + 1$, erit $\frac{z}{\int z} = \frac{3(4q+1)}{23q+5}$, & ob

ob $x = 16q + 4$ numeri primi eſſe debent $x - 1 = 16q + 3$ & $12x - 1 = 192q + 47$, hinc excluduntur valores $q = 3\alpha$.

q	$x - 1$	$12x - 1$	$\frac{z}{\int z}$
0	3	47	$\frac{3}{5}$ impoſſ.
1	19	239	$\frac{3\cdot5}{4\cdot7}$ $\boxed{\frac{5}{2\cdot3}}$ $\frac{3^2}{14}$ $\boxed{\frac{3^2}{13}}$ $\frac{13}{14}$; $z = 3^2\, 5.\, 13$ & numeri amicabiles erunt $\left\{\begin{matrix} 3^2\cdot 5\cdot 13\cdot 11\cdot 19 \\ 3^2\cdot 5\cdot 13\cdot 239 \end{matrix}\right)$
13	211	2543	$\frac{3\cdot 53}{16.19}$ $\boxed{\frac{53}{2\cdot 27}}$ $\frac{81}{8.19}$ $\boxed{\frac{243}{4.7.13}}$ $\frac{7.13}{2\cdot3\cdot19}$ $\boxed{\frac{13}{2.7}}$ $\frac{7^2}{3\cdot19}$ $\boxed{\frac{7^2}{3\cdot19}}$ Ergo $z = 3^5\cdot 7^2\cdot 13\cdot 53$ & numeri amicabiles erunt $\left\{\begin{matrix} 3^5.7^2.\, 13\cdot 53\cdot 11.211 \\ 3^5.7^2.\, 13\cdot 53\cdot 2543 \end{matrix}\right)$

Exemplum. 5.

§. CXIII. Sit $a = 5$; $b = 17$; & numeri amicabiles $5(3x - 1)z$ & $17(x - 1)z$; erit $\frac{z}{\int z} = \frac{18x}{32x - 22} = \frac{9x}{16x - 11}$;

 Cum

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Cùm x debeat eſſe numerus par, ponatur $x = 2p$, erit $\frac{z}{\int z} = \frac{18p}{32p - 11}$, & ex numeratore $18p$ vel factor 2 vel 3^2 tolli debet, ne ſit numerus redundans. At factor 2 tolli nequit; tollatur ergo factor 9. Ad hoc ponatur $p = 9q + 4$, ut ſit $x = 18q + 8$ & $x - 1 = 18q + 7$ & $3x - 1 = 54q + 23$, erit $\frac{z}{\int z} = \frac{2(9q+4)}{32q+13}$.

q	$x - 1$	$12x - 1$	$\frac{z}{\int z}$
0	7	23	$\frac{8}{13}$ imposſ.
2	43	131	$\frac{4 \cdot 11}{7 \cdot 11} = \frac{4}{7}$; $z = 4$ & N. A. $\begin{cases} 4 \cdot 5 \cdot 131 \\ 4 \cdot 17 \cdot 43 \end{cases}$
4	79	239	$\frac{16 \cdot 5}{3 \cdot 47}$
5	97	293	$\frac{2 \cdot 49}{173}$
17	313	941	$\frac{2 \cdot 157}{557}$
19	349	1049	$\frac{2 \cdot 5^2 \cdot 7}{27 \cdot 23}$
20	367	1103	$\frac{16 \cdot 23}{653}$
24	439	1319	$\frac{8 \cdot 5 \cdot 11}{781}$ inut. $= \frac{8 \cdot 5}{71}$

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Exempl. 6.

§. CXIV. Sit $a = 37$ & $b = 227$, erit $\int a = 38$, $\int b = 228$, & $\frac{m}{n} = \frac{1}{6}$; unde ſi numeri amicabiles ſint $37(6x-1)z$ & $227(x-1)z$, fiet $\frac{z}{\int z} = \frac{6.\,38x}{449x-264} = \frac{4.\,3.\,19x}{449x-264}$: Ubi cum x debeat eſſe numerus par, ponatut $x = 2p$, ut numeri primi eſſe debeant $x-1 = 2p-1$ & $6x-1 = 12p-1$, eritque $\frac{z}{\int z} = \frac{4.\,3.\,19p}{44p-132}$. Nunc ex numeratore vel factor 4 vel factor 3 tolli debet.

I. Tollatur factor 3, ad hoc ponatur $p = 3q$, ut ſit $\frac{z}{\int z} = \frac{4.\,3.\,19q}{449q-44}$; nunc fiat $q = 3r+1$, eritque;

$\frac{z}{\int z} = \frac{4.\,19(3r+1)}{449r+135}$. & $p = 9r+3$; $x-1 = 18r+5$; $6-1 = 108r+35$

r	$x-1$	$6x-1$	$\frac{z}{\int z}$
2	41	251	$\frac{4.\,19.\,7}{1033}$
3	59	359	$\frac{4.\,19.\,10}{1482} = \frac{4.\,5}{3.\,13}$
6	113	683	$\frac{4.\,19.\,19}{3.\,23.\,41}$
13	239	1439	$\frac{4.\,19.\,40}{4.\,1493}$

17	311	1871	$\frac{16.13.19}{8.971}$
22	401	2411	$\frac{4.19.67}{10013} = \frac{4.67}{17.31}$ $\left\lvert\frac{67}{4.17}\right\rvert$ $\frac{16}{31}$ $\left\lvert\frac{16}{31.}\right\rvert$
			$z = 16.67$
			Num. Amicab. $\left\{\begin{matrix} 16.67.37.2411 \\ 16.67.227.401 \end{matrix}\right)$
117	2111	12671	$\frac{4.19.352}{52668} = \frac{128.11.19}{4.7.9.11.19} = \frac{32}{63}$; $z = 32$
			& num. amic. $\left\{\begin{matrix} 32.37.12671 \\ 32.227.2111 \end{matrix}\right)$

II. Tollatur factor 4; ponatur $p = 4q$; erit $\frac{z}{\int z} = \frac{4.3.19q}{449q - 33}$ nunc ſit $q = 4r + 1$, erit $p = 16r + 4$; $x - 1 = 32r + 7$; $6x - 1 = 192r + 47$ atque $\frac{z}{\int z} = \frac{3.19(4r+1)}{449r + 104}$.

r	$x - 1$	$6x - 1$	$\frac{z}{\int z}$
0	7	47	$\frac{3.19}{8.13}$ $\left\lvert\frac{19}{4.5}\right\rvert$ $\frac{3.5}{2.13}$ $\left\lvert\frac{5}{2.3}\right\rvert$ $\frac{3^2}{13}$; $z = 3^2.5.19$
			& num. Am. $\left\{\begin{matrix} 3^2.5.19.37.47 \\ 3^2.5.19.227.7 \end{matrix}\right)$

2	71	431	$\frac{9.19}{2.167}$
8	263	1583	$\frac{3.19.33}{16.3.7.11} = \frac{3.19}{16.7} \Big\vert \frac{19}{4.5} \Big\vert \frac{3.5}{4.7} \Big\vert \frac{5}{2.3}$
			$\frac{3}{2.7} \Big\vert \frac{3^2}{13} \Big\vert \frac{13}{14}.$
			$z = 3^2.5.13.19$ &
			Num. Amic. $\begin{cases} 3^2.5.13.19.37.1583 \\ 3^2.5.13.19.227.263 \end{cases}$
15	487	2927	$\frac{3.19.61}{7.977}$
23	743	4463	$\frac{9.19.31}{9.19.61} = \frac{31}{61}.$
26	839	5039	$\frac{3.19.105}{2.3.13.151} = \frac{3.5.7.19}{2.13.151}$
30	967	5807	$\frac{3.19.11}{2.617}$
41	1319	7919	$\frac{3.19.165}{9.121.17} = \frac{5.19}{11.17}$

Exempl. 7.

§. CXV. Sit $a = 79$; $b = 11.19 = 209$; $\int a = 80$; $\int b = 240$ erit $m = 1, n = 3$, & numeri amicabiles ſint $79(3x - 1)z$ & $11.19(x - 1)z$, erit $\frac{z}{\int z} = \frac{240x}{446x - 288} = \frac{120x}{223x - 144}$. Sit

Sit $x = 2p$ erit $\frac{z}{\int z} = \frac{120p}{223p - 72}$, & numeri primi eſſe debent
$2p - 1$ & $6p - 1$

Nunc autem ex numeratore $120p$ vel factor 8 vel 3 tolli debet.

I. Tollatur factor 3; ſit $p = 9q$ erit $\frac{z}{\int z} = \frac{120q}{223q - 8}$ & fiat $q = 3r - 1$, ut ſit $\frac{z}{\int z} = \frac{40(3r - 1)}{223r - 77}$; $p = 27r - 9$ & $x - 1 = 54r - 19$; ac $3x - 1 = 162r - 55$.

Nunc autem ob 40 numerum redundantem vel 5 vel 4 tolli debet.

a) Tollatur 5, ſitque $r = 5s - 1$; erit $\frac{z}{\int z} = \frac{8(15s - 4)}{223s - 60}$ & numeros primos eſſe oportet $x - 1 = 470s - 73$; $3x - 1 = 810s - 217$. Ac ne ternarius denuo in numeratorem intret, excludendi ſunt caſus $s = 3\alpha - 1$. Hinc autem nihil invenitur.

c) Cum ſit $\frac{z}{\int z} = \frac{40(3r - 1)}{223r - 77}$ tollatur 4: ſitque $r = 4s - 1$; erit $\frac{z}{\int z} = \frac{10(12s - 1)}{223s - 75} = \frac{40(3s - 1)}{223s - 75}$; ſit porro $s = 4t + 1$ erit $\frac{z}{\int z} = \frac{10(12t + 2)}{223t + 37} = \frac{20(6t + 1)}{223t + 37}$. Sit porro $t = 2u - 1$; erit $\frac{z}{\int z} = \frac{10(12u - 5)}{223u - 93}$; & ob $r = 16t + 3 = 32u - 13$; erit

$$x - 1 = 1728u - 721$$
$$3x - 1 = 5184u - 2161$$

At hos numeros non reddit primos minor valor ipſius u quam 16, unde ſit $\frac{z}{\int z} = \frac{2.11.17}{5.139}$, qui ob factorem 11 eſt inutilis.

II. Er-

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II. Ergo ex æq. $\frac{z}{\int z} = \frac{120q}{223p-72}$ tollatur factor 8. Ponatur $p = 8q$, erit $\frac{z}{\int z} = \frac{120q}{223q-9}$ & nunc ſit $q = 8r - 1$ erit $\frac{z}{\int z} = \frac{3.5(8r-1)}{223r-29}$; at ob $p = 64r - 8$, erit

$$x - 1 = 128r - 17;\; 3x - 1 = 384r - 49$$

Unde valores exluduntur $r = 3\alpha + 1$; & $r = 5\alpha \pm 1$.

r	$x-1$	$3x-1$	$\frac{z}{\int z}$
2	239	719	$\frac{3.5^2}{139}$
3	367	1103	$\frac{3.23}{128}$ $\frac{23}{8.3}$ $\frac{3^2}{16}$ $\frac{3^2}{13}$ $\frac{13}{16}$ $\frac{13}{14}$ $\frac{7}{8}$
			Ergo $z = 3^2.7.13.23$ vel
			$\frac{3.23}{128}$ $\frac{23}{8.3}$ $\frac{3^2}{16}$ $\frac{3^3}{8.5}$ $\frac{5}{6}$; ergo $z =$

$3^3.5.23$ & numeri amicabiles erunt.

$$\left\{\begin{array}{l} 3^2.7.13.23.79.1103 \\ 3^2.7.13.23.11.19.367 \end{array}\right\} \text{ vel } \left\{\begin{array}{l} 3^3.5.23.79.1103 \\ 3^3.5.23.11.19.367 \end{array}\right\}$$

Exempl. 8.

§. CXVI. Sit $a = 17.19$: $b = 11.59$; erit $\int a = 18.20$, $\int b =$

$\int b = 12.60$, & $m = 1$, $n = 2$. Si ergo num. am. ponantur.
$\begin{matrix} 17.19(2x-1)z \\ 11.59(x-1)z \end{matrix}$ erit $\frac{z}{\int z} = \frac{720x}{1295x-972}$: Sit $x = 2p$ erit $\frac{z}{\int z} = \frac{720p}{1295p-486}$; atque $\begin{matrix} x-1 = 2p-1 \\ 2x-1 = 4p-1 \end{matrix}$; quorum ut neuter ſit diviſibilis per 3, debet eſſe $p = 3q$, ut ſit

$$\frac{z}{\int z} = \frac{720q}{1295q-162}; \quad \& \quad \begin{matrix} x-1 = 6q-1 \\ 2x-1 = 12q-1 \end{matrix}$$

Tollatur ex num. factor 16, ſitque $q = 2r$ erit

$$\frac{z}{\int z} = \frac{720r}{1295r-81}; \text{ nunc ſit } r = 16s-1 \text{ erit}$$

$$\frac{z}{\int z} = \frac{45(16s-1)}{1295s-86} \quad \& \quad \begin{matrix} x-1 = 192s-13 \\ 2x-1 = 384s-25 \end{matrix}$$

Sit $s = 1$; erit $x-1 = 179$; $2x-1 = 359$ &

$$\frac{z}{\int z} = \frac{45.15}{1209} = \frac{225}{403} = \frac{3^2.\ 5^2.}{13.\ 31} \quad \boxed{\frac{3^2}{13}} \quad \frac{5^2}{31} \quad \boxed{\frac{5^2}{31}}$$

Ergo $z = 3^2.5^2$ & numeri amicabiles erunt

$$\left\{ \begin{matrix} 3^2.5^2.17.19.359 \\ 3^2.5^2.11.59.179 \end{matrix} \right)$$

Scholion.

§. CXVII. Hæc ultima methodus in problemate 5. expoſita prorſus diverſa eſt a methodo præcedente, quam problemata 4 priora complectuntur: dum in hac factor communis quæritur, in illa autem datur. Utraque tamen ſingulari præſtantiæ genere eſt prædita, ut altera ſine ſubſidio alterius non ſatis apta ſit ad multitudinem

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tudinem numerorum amicabilium augendam. Poſterior enim methodus ſuppeditat ejusmodi factores communes, quos ad uſum prioris vix ſuſpicari licuiſſet: prior vero ſuggerit reliquos factores huic inſtituto idoneos. Ceterum cuncta, quae hic tradidi, ſpecimen continent methodi ſummae incertae, quam, quantum licuit, ad regulas algebraicas reduxi, ut vaga tentandi incertitudo reſtringeretur. Coronidis ergo loco ultra ſexaginta numerorum amicabilium paria ſubjungam, quos his methodis elicui.

Catalogus numerorum amicabilium.

I $\begin{cases} 2^2 \cdot 5 \cdot 11 \\ 2^2 \cdot 71 \end{cases}$ II $\begin{cases} 2^4 \cdot 23 \cdot 47 \\ 2^4 \cdot 1151 \end{cases}$ III $\begin{cases} 2^7 \cdot 191 \cdot 383 \\ 2^7 \cdot 73727 \end{cases}$

IV $\begin{cases} 2^2 \cdot 23 \cdot 5 \cdot 137 \\ 2^2 \cdot 23 \cdot 827 \end{cases}$ V $\begin{cases} 3^2 \cdot 7 \cdot 13 \cdot 5 \cdot 17 \\ 3^2 \cdot 7 \cdot 13 \cdot 107 \end{cases}$

VI $\begin{cases} 3^2 \cdot 5 \cdot 13 \cdot 11 \cdot 19 \\ 3^2 \cdot 5 \cdot 13 \cdot 239 \end{cases}$ VII $\begin{cases} 3^2 \cdot 7^2 \cdot 13 \cdot 5 \cdot 41 \\ 3^2 \cdot 7^2 \cdot 13 \cdot 251 \end{cases}$

VIII $\begin{cases} 3^2 \cdot 5 \cdot 7 \cdot 53 \cdot 1889 \\ 3^2 \cdot 5 \cdot 7 \cdot 102059 \end{cases}$ IX $\begin{cases} 2^2 \cdot 13 \cdot 17 \cdot 389 \cdot 509 \\ 2^2 \cdot 13 \cdot 17 \cdot 198899 \end{cases}$

X $\begin{cases} 3^2 \cdot 5 \cdot 19 \cdot 37 \cdot 7 \cdot 887 \\ 3^2 \cdot 5 \cdot 19 \cdot 37 \cdot 7103 \end{cases}$ XI $\begin{cases} 3^4 \cdot 5 \cdot 11 \cdot 29 \cdot 89 \\ 3^4 \cdot 5 \cdot 11 \cdot 2699 \end{cases}$

XII $\begin{cases} 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 41 \cdot 461 \\ 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19403 \end{cases}$ XIII $\begin{cases} 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 29 \cdot 569 \\ 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 17099 \end{cases}$

XIV $\begin{cases} 3^2 \cdot 7^2 \cdot 13 \cdot 97 \cdot 5 \cdot 193 \\ 3^2 \cdot 7^2 \cdot 13 \cdot 97 \cdot 1163 \end{cases}$ XV $\begin{cases} 3^2 \cdot 7 \cdot 13 \cdot 41 \cdot 163 \cdot 5 \cdot 977 \\ 3^2 \cdot 7 \cdot 13 \cdot 41 \cdot 163 \cdot 5867 \end{cases}$

XVI $\begin{cases} 2^3 \cdot 17 \cdot 79 \\ 2^3 \cdot 23 \cdot 59 \end{cases}$ XVII $\begin{cases} 2^4 \cdot 23 \cdot 1367 \\ 2^4 \cdot 53 \cdot 607 \end{cases}$

XVIII $\begin{cases} 2^4 \cdot 47 \cdot 89 \\ 2^4 \cdot 53 \cdot 79 \end{cases}$ XIX $\begin{cases} 2^4 \cdot 23 \cdot 479 \\ 2^4 \cdot 89 \cdot 127 \end{cases}$

XX	$2^4 \cdot 23 \cdot$ [illegible] 467	$2^4 \cdot$ [illegible]
XXI	$2^4 \cdot 17 \cdot$ [illegible]	$2^4 \cdot$ [illegible] $\cdot 383$
XXII	$2^4 \cdot 17 \cdot 10303$	$2^4 \cdot 167 \cdot 1103$
XXIII	$2^4 \cdot 19 \cdot$ [illegible] 439	$2^4 \cdot$ [illegible] 49 $\cdot$ [illegible] 91
XXIV	$2^5 \cdot 59 \cdot 1103$	$2^5 \cdot 79 \cdot 827$
XXV	$2^5 \cdot 37 \cdot 12671$	$2^5 \cdot 227 \cdot 2111$
XXVI	$2^5 \cdot 53 \cdot 10559$	$2^5 \cdot 79 \cdot 7127$
XXVII	$2^6 \cdot 79 \cdot 11087$	$2^6 \cdot 383 \cdot 2309$
XXVIII	$2^8 \cdot 383 \cdot 9203$	$2^8 \cdot 1151 \cdot 3067$
XXIX	$2^2 \cdot 11 \cdot 17 \cdot 263$	$2^2 \cdot 11 \cdot 43 \cdot 107$
XXX	$3^2 \cdot 5 \cdot 7 \cdot 71$	$3^2 \cdot 5 \cdot 17 \cdot 31$
XXXI	$3^2 \cdot 5 \cdot 13 \cdot 29 \cdot 79$	$3^2 \cdot 5 \cdot 13 \cdot 11 \cdot 199$
XXXII	$3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 47$	$3^2 \cdot 5 \cdot 13 \cdot 29 \cdot 31$
XXXIII	$3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 37 \cdot 1583$	$3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 227 \cdot 263$
XXXIV	$3^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 11 \cdot 220499$	$3^2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 89 \cdot 29399$
XXXV	$3^2 \cdot 5 \cdot 19 \cdot 37 \cdot 47$	$3^2 \cdot 5 \cdot 19 \cdot 7 \cdot 227$
XXXVI	$2^4 \cdot 67 \cdot 37 \cdot 2411$	$2^4 \cdot 67 \cdot 227 \cdot 401$
XXXVII	$3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 29$	$3^3 \cdot 5 \cdot 31 \cdot 89$
XXXVIII	$2 \cdot 5 \cdot 23 \cdot 29 \cdot 673$	$2 \cdot 5 \cdot 7 \cdot 60659$
XXXIX	$2 \cdot 5 \cdot 7 \cdot 19 \cdot 107$	$2 \cdot 5 \cdot 47 \cdot 359$
XL	$2^3 \cdot 11 \cdot 163 \cdot 191$	$2^3 \cdot 31 \cdot 11807$
XLI	$3^2 \cdot 7 \cdot 13 \cdot 23 \cdot 11 \cdot 19 \cdot 367$	$3^2 \cdot 7 \cdot 13 \cdot 23 \cdot 79 \cdot 1103$
XLII	$3^2 \cdot 5 \cdot 23 \cdot 11 \cdot 19 \cdot 367$	$3^2 \cdot 5 \cdot 23 \cdot 79 \cdot 1103$
XLIII	$2^3 \cdot 11 \cdot 59 \cdot 173$	$2^3 \cdot 57 \cdot 2609$
XLIV	$2^3 \cdot 11 \cdot 23 \cdot 2543$	$2^3 \cdot 383 \cdot 1907$
XLV	$2^3 \cdot 11 \cdot 23 \cdot 1871$	$2^3 \cdot 467 \cdot 1151$
XLVI	$2^3 \cdot 11 \cdot 23 \cdot 1619$	$2^3 \cdot 719 \cdot 647$
XLVII	$2^3 \cdot 11 \cdot 29 \cdot 239$	$2^3 \cdot 191 \cdot 449$

XLVIII

XLVIII $\begin{cases} 2^3.29.47.59 \\ 2^3.17.4799 \end{cases}$ XLIX $\begin{cases} 2^4.17.167.13679 \\ 2^4.809.51071 \end{cases}$

L $\begin{cases} 2^4.23.47.9767 \\ 2^4.1583.7103 \end{cases}$ LI $\begin{cases} 2^2.5.13.1187 \\ 2^2.43.2267 \end{cases}$

LII $\begin{cases} 3^2.7.13.5.17.1187 \\ 3^2.7.13.131.971 \end{cases}$ LIII $\begin{cases} 3^2.7^2.13.53.11.211 \\ 3^2.7^2.13.53.2543 \end{cases}$

LIV $\begin{cases} 3^2.5^2.11.59.179 \\ 3^2.5^2.17.19.359 \end{cases}$ LV $\begin{cases} 3^3.5.17.23.397 \\ 3^3.5.7.21491 \end{cases}$

LVI $\begin{cases} 3^4.7.11^2.19.47.7019 \\ 3^4.7.11^2.19.389.863 \end{cases}$ LVII $\begin{cases} 3^4.7.11^2.19.53.6959 \\ 3^4.7.11^2.19.179.2087 \end{cases}$

LVIII $\begin{cases} 3^5.7^2.13.19.47.7019 \\ 3^5.7^2.13.19.389.863 \end{cases}$ LIX $\begin{cases} 3^5.7^2.13.19.53.6959 \\ 3^5.7^2.13.19.179.2087 \end{cases}$

His adjicere lubet duo paria ſequentia, quæ ſunt formæ diverſæ à præcedentibus:

LX $\begin{cases} 2^3.19.41 \\ 2^5.199 \end{cases}$ LXI $\begin{cases} 2^3.41.467 \\ 2^5.19.233 \end{cases}$

Demonſtratio Gemina THEOREMATIS NEUTONIANI quo traditur relatio inter coëfficientes cujusvis æquationis algebraicæ & ſummas poteſtatum radicum ejusdem.

§. I.

Poſtquam æquatio algebraica tam a fractionibus quam ab irrationalitate fuerit liberata, atque ad hujusmodi formam reducta:

$$x^{n} - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \dots + N = 0$$

demonſtrari ſolet in analyſi, hujusmodi æquationem tot ſemper habere radices, ſive ſint reales ſive imaginariæ, quot unitates contineantur in poteſtatis ſummæ exponente n. Tum vero non minus certum eſt, ſi hujus æquationis omnes radices fuerint α, β, γ, δ, ε, - - ν, coëfficientes terminorum æquationis A, B, C, D, E, &c. ex his radicibus ita conflari, ut ſit:

A = ſummæ omnium radicum $= \alpha + \beta + \gamma + \delta + \dots + \nu$

B = ſummæ productorum ex binis $= \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma +$ &c.

C = ſummæ productorum ex ternis $= \alpha\beta\gamma +$ &c.

D = ſummæ productorum ex quaternis $= \alpha\beta\gamma\delta +$ &c.

E = ſummæ productorum ex quinis $= \alpha\beta\gamma\delta\varepsilon +$ &c.

&c.

Ultimumque tandem terminum abſolutum $\pm$ N eſſe productum ex omnibus radicibus $\alpha\beta\gamma\delta \dots \nu$.

§. II.

§. II. Quo jam theorema, cujus demonſtrationem hic tradere conſtitui, facilius ac brevius enunciare queam; deſignet $\int \alpha$ ſummam omnium radicum; $\int \alpha^2$ ſummam quadratorum earumdem radicum; $\int \alpha^3$ ſummam cuborum radicum; $\int \alpha^4$ ſummam biquadratorum iſtarum radicum, & ita porro: ita ut ſit:

$$\int \alpha = \alpha + \beta + \gamma + \delta + \varepsilon + \ldots + \nu$$

$$\int \alpha^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 + \ldots + \nu^2$$

$$\int \alpha^3 = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \varepsilon^3 + \ldots + \nu^3$$

$$\int \alpha^4 = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \varepsilon^4 + \ldots + \nu^4$$

$$\int \alpha^5 = \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \varepsilon^5 + \ldots + \nu^5$$

&c.

§. III. Hac ſignandi ratione expoſita Neutonus affirmat iſtas poteſtatum, quæ ex ſingulis radicibus formantur, ſummas per coëfficientes æquationis A, B, C, D, E, &c. ita definiri, ut ſit.

$$\int \alpha = A$$

$$\int \alpha^2 = A\int \alpha - 2B$$

$$\int \alpha^3 = A\int \alpha^2 - B\int \alpha + 3C$$

$$\int \alpha^4 = A\int \alpha^3 - B\int \alpha^2 + C\int \alpha - 4D$$

$$\int \alpha^5 = A\int \alpha^4 - B\int \alpha^3 + C\int \alpha^2 - D\int \alpha + 5E$$

$$\int \alpha^6 = A\int \alpha^5 - B\int \alpha^4 + C\int \alpha^3 - D\int \alpha^2 + E\int \alpha - 6F$$

&c.

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Cujus theorematis demonſtrationem Neutonus non ſolum nullam tradit, ſed etiam Ipſe videtur ejus veritatem ex continua illatione concluſiſſe. Primum enim demonſtratione non eget eſſe $\int\alpha = A$: & cum ſit

$$A^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 + \&c. + 2\alpha\beta + 2\alpha\gamma + 2\alpha\delta + 2\beta\gamma + 2\beta\delta + \&c.$$

erit $A^2 = \int\alpha^2 + 2B$, ideoque $\int\alpha^2 = A^2 - 2B = A\int\alpha - 2B$: ſimilique modo veritas ſequentium formularum evinci poteſt; ſed continuo majore opus erit labore.

§. IV. Cum plures jam hujus theorematis utiliſſimi veritatem oſtenderint, eorum demonſtrationes autem regulis combinationum plerumque innitantur, quæ etiamſi veræ ſint, tamen ab inductione plurimum pendeant, duplicem hic afferam demonſtrationem, in quarum utraque inductioni nihil tribuitur. Altera quidem ex analyſi infinitorum eſt petita, quæ etſi nimis longe remota videatur, tamen totum negotium perfecte conficit: verum tamen cum contra eam jure objici queat, hujus theorematis veritatem evictam eſſe oportere, antequam ad analyſin infinitorum perveniatur; alteram demonſtrationem adjungam, in qua nihil aſſumitur, niſi quod ſtatim ab initio in explicatione naturæ æquationum tradi ſolet.

Demonſtratio I.

§. V. Ponatur $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \ldots \pm N = Z$ & cum æquationis $Z = 0$ radices ſeu valores ipſius x ſint, $\alpha, \beta, \gamma, \delta, \ldots \nu$, quorum numerus eſt $= n$, erit ex natura æquationum:

$$Z = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\ldots(x-\nu)$$

et

et logarithmis ſumendis habebitur:

$$lZ = l(x-\alpha)+l(x-\beta)+l(x-\gamma)+l(x-\delta)+\&c.$$
$$+\,l(x-\nu)$$

Quod ſi jam harum formularum differentialia capiantur erit:

$$\frac{dZ}{Z} = \frac{dx}{x-\alpha}+\frac{dx}{x-\beta}+\frac{dx}{x-\gamma}+\frac{dx}{x-\delta}+\&c.$$
$$+\,\frac{dx}{x-\nu}$$

Ideoque per dx dividendo fiet:

$$\frac{dZ}{Zdx} = \frac{1}{x-\alpha}+\frac{1}{x-\beta}+\frac{1}{x-\gamma}+\frac{1}{x-\delta}+\cdots+\frac{1}{x-\nu}$$

Convertantur nunc ſingulæ hæ fractiones more ſolito in ſeries geometricas infinitas: ob

$$\frac{1}{x-\alpha} = \frac{1}{x}+\frac{\alpha}{x^2}+\frac{\alpha^2}{x^3}+\frac{\alpha^3}{x^4}+\frac{\alpha^4}{x^5}+\frac{\alpha^5}{x^6}+\&c.$$

$$\frac{1}{x-\beta} = \frac{1}{x}+\frac{\beta}{x^2}+\frac{\beta^2}{x^3}+\frac{\beta^3}{x^4}+\frac{\beta^4}{x^5}+\frac{\beta^5}{x^6}+\&c.$$

$$\frac{1}{x-\gamma} = \frac{1}{x}+\frac{\gamma}{x^2}+\frac{\gamma^2}{x^3}+\frac{\gamma^3}{x^4}+\frac{\gamma^4}{x^5}+\frac{\gamma^5}{x^6}+\&c.$$

&c.

$$\frac{1}{x-\nu} = \frac{1}{x}+\frac{\nu}{x^2}+\frac{\nu^2}{x^3}+\frac{\nu^3}{x^4}+\frac{\nu^4}{x^5}+\frac{\nu^5}{x^6}+\&c.$$

His

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His igitur ſeriebus colligendis, ſignisque ante expoſitis: $\int a$, $\int a^2$, $\int a^3$ &c. introducendis invenietur, quia numerus harum ſerierum eſt $= n$:

$$\frac{dZ}{Zdx} = \frac{n}{x} + \frac{1}{x^2}\int a + \frac{1}{x^3}\int a^2 + \frac{1}{x^4}\int a^3 + \frac{1}{x^5}\int a^4 + \&c.$$

§. VI. Cum autem ſtatuerimus:

$$Z = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} \ldots + N$$

erit ſimiliter differentialibus ſumendis:

$$\frac{dZ}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \&c.$$

hincque colligetur ſuperior formula $\frac{dZ}{Zdx}$ ita expreſſa ut ſit:

$$\frac{dZ}{Zdx} = \frac{nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \&c.}{x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c.}$$

quæ igitur fractio æqualis eſſe debet ſeriei ſupra inventæ:

$$\frac{n}{x} + \frac{1}{x^2}\int a + \frac{1}{x^3}\int a^2 + \frac{1}{x^4}\int a^3 + \frac{1}{x^5}\int a^4 + \&c.$$

Quare

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Quare ſi utraque expreſſio pro $\frac{dZ}{Zdx}$ inventa per alterius denominatorem $x^{n} - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} + \&c.$ multiplicetur reſultabit hæc æquatio:

$$
\begin{aligned}
& nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \&c. \\
= \; & nx^{n-1} + x^{n-2}\int\alpha + x^{n-3}\int\alpha^{2} + x^{n-4}\int\alpha^{3} + x^{n-5}\int\alpha^{4} + \&c. \\
& - nAx^{n-2} - Ax^{n-3}\int\alpha - Ax^{n-4}\int\alpha^{2} - Ax^{n-5}\int\alpha^{3} - \&c. \\
& + nBx^{n-3} + Bx^{n-4}\int\alpha + Bx^{n-5}\int\alpha^{2} + \&c. \\
& - nCx^{n-4} - Cx^{n-5}\int\alpha - \&c. \\
& + nDx^{n-5} + \&c.
\end{aligned}
$$

§. VII. Quemadmodum jam utrinque termini primi nx^{n-1} ſunt æquales, neceſſe eſt ut & ſecundi, tertii, quarti &c. inter ſe ſeorſim æquentur; unde ſequentes naſcentur æquationes:

$$
\begin{aligned}
& -(n-1)A = \int\alpha - nA \\
& +(n-2)B = \int\alpha^{2} - A\int\alpha + nB \\
& -(n-3)C = \int\alpha^{3} - A\int\alpha^{2} + B\int\alpha - nC \\
& +(n-4)D = \int\alpha^{4} - A\int\alpha^{3} + B\int\alpha^{2} - C\int\alpha + nD \\
& \qquad \&c.
\end{aligned}
$$

harumque æquationum lex, qua progrediuntur, ſponte eſt

manifeſta. Ex iis autem obtinentur formulæ illæ ipſæ, quibus theorema Neutonianum conſtat; ſcilicet:

$$\int\alpha = A$$
$$\int\alpha^2 = A\int\alpha - 2B$$
$$\int\alpha^3 = A\int\alpha^2 - B\int\alpha + 3C$$
$$\int\alpha^4 = A\int\alpha^3 - B\int\alpha^2 + C\int\alpha - 4D$$
$$\int\alpha^5 = A\int\mu^4 - B\int\alpha^3 + C\int\alpha^2 - D\int\alpha + 5E$$

Quæ eſt altera theorematis propoſiti demonſtratio.

Demonſtratio II.

§. VIII. Quo hujus demonſtrationis vis clarius perſpiciatur, eam ad æquationem determinati gradus accommodabo, ita tamen ut ea intelligatur ad quosvis gradus æque patere. Sit ergo propoſita æquatio quinti gradus:

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

cujus quinque radices ſint α, β, γ, δ, ε. Quia igitur quælibet radix loco x ſubſtituta æquationi ſatisfacit, erit:

$$\alpha^5 - A\alpha^4 + B\alpha^3 - C\alpha^2 + D\alpha - E = 0$$
$$\beta^5 - A\beta^4 + B\beta^3 - C\beta^2 + D\beta - E = 0$$
$$\gamma^5 - A\gamma^4 + B\gamma^3 - C\gamma^2 + D\gamma - E = 0$$
$$\delta^5 - B\delta^4 + B\delta^3 - C\delta^2 + D\delta - E = 0$$
$$\varepsilon^5 - A\varepsilon^4 + B\varepsilon^3 - C\varepsilon^2 + D\varepsilon - E = 0$$

Colligentur hæ æquationes in unam ſummam, & ob ſigna ſupra recepta (§. 2.) habebitur:

$$\int\alpha^5 - A\int\alpha^4 + B\int\alpha - C\int\alpha^2 + D\int\alpha - 5E = 0$$
$$\text{ſeu } \int\alpha^5 = A\int\alpha^4 - B\int\alpha^3 + C\int\alpha^2 - D\int\alpha + 5E.$$

§. IX.

§. IX. Hinc dilucide patet, ſi æquatio propoſita fuerit gradus cujuscunque

$$x^{n} - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \ldots\ldots \pm Mx \mp N = 0$$

ubi in ultimis terminis ſignorum ambiguorum ſuperiora valent, ſi exponens ſummus n fuerit numerus impar, inferiora ſi par; fore pariter:

$$ſ\alpha^{n} = Aſ\alpha^{n-1} - Bſ\alpha^{n-2} + Cſ\alpha^{n-3} - \ldots\ldots \mp Mſ\alpha \pm nN$$

ſi quidem per α indicetur radix quælibet iſtius æquationis ſicque veritas Theorematis Neutoniani jam pro uno caſu eſt oſtenſa. Super eſt igitur, ut ejusdem veritatem tam pro altioribus quam pro inferioribus radicum poteſtatibus demonſtremus.

§. X. Pro altioribus quidem poteſtatibus res pari modo patet, ſi enim valores $\alpha, \beta, \gamma, \delta, \varepsilon$ ſatisfaciant æquationi

$$x^{5} - Ax^{4} + Bx^{3} - Cx^{2} + Dx - E = 0$$

ſatisfacient quoque ſequentibus æquationibus:

$$x^{6} - Ax^{5} + Bx^{4} - Cx^{3} + Dx^{2} - Ex = 0$$
$$x^{7} - Ax^{6} + Bx^{5} - Cx^{4} + Dx^{3} - Ex^{2} = 0$$
$$x^{8} - Ax^{7} + Bx^{6} - Cx^{5} + Dx^{4} - Ex^{3} = 0$$

&c.

Ac propterea ſi in unaquaque æquatione pro x ſinguli valores $\alpha, \beta, \gamma, \delta, \varepsilon$ ſubſtituantur, & aggregata colligantur, erit

$$\int\alpha^{6} = A\int\alpha^{5} - B\int\alpha^{4} + C\int\alpha^{3} - D\int\alpha^{2} + E\int\alpha$$

$$\int\alpha^{7} = A\int\alpha^{6} - B\int\alpha^{5} + C\int\alpha^{4} - D\int\alpha + E\int\alpha^{2}$$

$$\int\alpha^{8} = A\int\alpha^{7} - B\int\alpha^{6} + C\int\alpha^{5} - D\int\alpha^{4} + E\int\alpha^{3}$$

&c.

§. XI. Si ergo α denotet radicem quamcunque hujus æquationis:

$$x^{n} - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{----} + Mx \mp N = 0$$

erit non ſolum, uti jam invenimus:

$$\int\alpha^{n} = A\int\alpha^{n-1} - B\int\alpha^{n-2} + C\int\alpha^{n-3} - D\int\alpha^{n-4} \text{----} \mp M\int\alpha \pm nN$$

ſed etiam ad altiores quoque poteſtates progrediendo erit:

$$\int\alpha^{n+1} = A\int\alpha^{n} - B\int\alpha^{n-2} + C\int\alpha^{n-3} - D\int\alpha^{n-4} \text{----} \mp M\int\alpha^{2} \pm N\int\alpha$$

$$\int\alpha^{n+2} = A\int\alpha^{n+1} - B\int\alpha^{n} + C\int\alpha^{n-1} - D\int\alpha^{n-2} + \text{----} \mp M\int\alpha^{3} \pm N\int\alpha^{2}$$

$$\int\alpha^{n+3} = A\int\alpha^{n+2} - B\int\alpha^{n+1} + C\int\alpha^{n} - D\int\alpha^{n-1} + \text{---} \mp M\int\alpha^{4} \pm N\int\alpha^{3}$$

&c.

& in genere quidem, ſi ad n addatur numerus quicunque m, erit

$$\int\alpha^{n+m} = A\int\alpha^{n+m-1} - B\int\alpha^{n+m-2} + C\int\alpha^{n+m-3} \text{-----} \mp M\int\alpha^{m+1} \pm N\int\alpha^{m}.$$

Ubi quidem notandum eſt, ſi ſit $m = 0$, ob ſingulas poteſtates $\alpha^{0} = 1$, $\beta^{0} = 1$, $\gamma^{0} = 1$; &c. numerumque harum litterarum $= n$ fore

fore $\int\alpha = n$, quo caſu formula primo inventa in hac expreſſione continetur.

§. XII. Quamquam autem hæc expreſſio æque veritati eſt conſentanea, ſi pro m accipiatur numerus negativus: hincque pro æquatione quinti gradus aſſumta

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

ſequentes formulæ pariter locum habent:

$$\int\alpha^4 = A\int\alpha^3 - B\int\alpha^2 + C\int\alpha^1 - D\int\alpha^0 + E\int\alpha^{-1}$$
$$\int\alpha^3 = A\int\alpha^2 - B\int\alpha^1 + C\int\alpha^0 - D\int\alpha^{-1} + E\int\alpha^{-2}$$
$$\int\alpha^2 = A\int\alpha^1 - B\int\alpha^0 + C\int\alpha^{-1} - D\int\alpha^{-2} + E\int\alpha^{-3}$$
&c.

tamen hæ formulæ ſunt diverſæ ab iis, quas theorema continet. Demonſtrandum enim eſt eſſe:

$$\int\alpha^4 = A\int\alpha^3 - B\int\alpha^2 + C\int\alpha - 4D$$
$$\int\alpha^3 = A\int\alpha^2 - B\int\alpha + 3C$$
$$\int\alpha^2 = A\int\alpha - 2B$$
$$\int\alpha = A$$

Harum igitur formularum veritatem ſequenti modo oſtendo.

§. XIII. Propoſita ſcilicet æquatione quinti gradus:

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0.$$

 For-

Formentur retinendis iisdem coëfficientibus ſequentes æquationes inferiorum graduum:

I. $x - A = o.$ Radix ſit p

II. $x^2 - Ax + B = o.$ Radix quælibet ſit q

III. $x^3 - Ax^2 + Bx - C = o.$ Sit radix quælibet r

IV. $x^4 - Ax^3 + Bx^2 - Cx + D = o.$ Radix quælibet s

Quarum æquationum radices, etiamſi inter ſe maxime diſcrepent, tamen in his ſingulis æquationibus eandem conſtituent ſummam $= A$. Deinde remota prima ſumma productorum ex binis radicibus ubique erit eadem $= B$: Tum ſumma productorum ex ternis radicibus ubique erit $= C$, præter æquationes ſcilicet I & II, ubi C non occurrit. Similiter in IV & propoſita ſumma productorum ex quaternis radicibus erit eadem $= D$.

§. XIV. In quibus autem æquationibus non ſolum ſumma radicum eſt eadem, ſed etiam ſumma productorum ex binis radicibus, ibi quoque ſumma quadratorum radicum eſt eadem. Sin autem præterea ſumma productorum ex ternis radicibus fuerit eadem, tum ſumma quoque cuborum omnium radicum erit eadem. Atque ſi inſuper ſumma productorum ex quaternis radicibus fuerit eadem, tum quoque ſumma biquadratorum omnium radicum erit eadem atque ita porro. Hic ſcilicet aſſumo, quod facile concedetur, ſummam quadratorum per ſummam radicum & ſummam productorum ex binis determinari; ſummam cuborum autem præterea requirere ſummam factorum ex ternis radicibus; ac ſummam biquadratorum præterea ſummam factorum ex quaternis

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ternis radicibus, & ita porro; quod quidem demonſtratu non eſſet difficile.

§. XV. In æquationibus ergo inferiorum graduum, quarum radices denotantur reſpective per litteras p, q, r, s, dum ipſius propoſitæ quinti gradus quælibet radix littera u indicatur, erit:

$$\int u = \int s = \int r = \int q = \int p$$

$$\int u^2 = \int s^2 = \int r^2 = \int q^2$$

$$\int u^3 = \int s^3 = \int r^3$$

$$\int u^4 = \int s^4$$

At per ea quæ ante §. 9 demonſtravimus eſt

$$\int p = A$$

$$\int q^2 = A\int q - 2B$$

$$\int r^3 = A\int r^2 - B\int r + 3C$$

$$\int s^4 = A\int s^3 - B\int s^2 + C\int s - 4D$$

Hinc ergo nanciſcimur pro æquatione quinti gradus propoſita: $x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$ has formulas

ſa

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$$ſ\alpha = A$$

$$ſ\alpha^2 = Aſ\alpha - 2B$$

$$ſ\alpha^3 = Aſ\alpha^2 - Bſ\alpha + 3C$$

$$ſ\alpha^4 = Aſ\alpha^3 - Bſ\alpha^2 + Cſ\alpha - 4D$$

§. XVI. In æquatione ergo cujuscunque gradus propoſita:

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. \cdots \pm N = 0$$

ſi quælibet radix littera α indicetur erit:

$$ſ\alpha = A$$

$$ſ\alpha^2 = Aſ\alpha - 2B$$

$$ſ\alpha^3 = Aſ\alpha^2 - Bſ\alpha + 3C$$

$$ſ\alpha^4 = Aſ\alpha^3 - Bſ\alpha^2 + Cſ\alpha - 4D$$

$$ſ\alpha^5 = Aſ\alpha^4 - Bſ\alpha^3 + Cſ\alpha^2 - Dſ\alpha + 5E$$

&c.

Hocque modo veritas Theorematis Neutoniani pariter habetur demonſtrata.

Ani-

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Animadverſiones
in
Rectificationem Ellipſis.

§. I.

Tab. I.

Ellipſis rectificatio tot jam variis methodis eſt fruſtra tentata, ut non ſolum comparationem arcuum ellipticorum cum lineis rectis, ſed etiam ne cum circularibus quidem aut parabolicis expectare nequeamus. Cum enim formula illa differentialis, cujus integrale arcum ellipticum indefinitum exprimit, nullo modo ab irrationalitate liberari queat; certum hoc eſt ſignum, ejus integrationem non ſolum non algebraice, ſed etiam ne conceſſis quidem circuli & hyperbolæ quadraturis perfici poſſe. Quod cum tenendum ſit de rectificatione ellipſis indefinita, hinc adhuc minime ſequitur, arcum quempiam definitum veluti totam perimetrum ellipſis omnem comparationem cum lineis vel rectis vel circularibus penitus reſpuere: propterea quod jam innumerabiles curvæ aſſignari poſſunt indefinite æque parum rectificabiles atque ellipſis, in quibus tamen arcus definiti per lineas rectas menſurari queant.

§. II. Miſſa igitur rectificatione ellipſis indefinita, definitam potius ſum aggreſſus, experturus, utrum tota cujusque ellipſis perimeter non commode poſſit ad menſuras cognitas, quorſum etiam logarithmos & arcus circulares refero, per expreſſiones finitas revocari. Quanquam autem in hac inveſtigatione nihil admodum ſum conſecutus, quod ſcopo meo ſatisfeciſſet; tamen præter expectationem nonnulla ſe mihi obtulerunt phænomena

ſatis ſingularia, quibus theoria linearum curvarum non mediocriter promoveri videtur. Tum vero etiam difficultates, quæ in toto hoc calculo occurrerunt, anſam mihi præbuerunt quædam inſignia artificia inveniendi, quæ tam in calculo integrali, quam in theoria ſerierum infinitarum ingentem utilitatem ſæpius afferre poſſe videntur. Quamobrem operæ pretium fore exiſtimavi, ſi has ſpeculationes totumque quaſi filum calculorum meorum dilucide expoſuero.

Propoſitio.

Fig. 1. *§. III. Super data recta AC tanquam altero ſemiaxe deſcriptos concipio infinitos quadrantes ellipticos AP, AB, Ap, quorum ergo omnium eſt centrum C; alteri vero ſemiaxes conjugati ſunt CP, CB, Cp. Tum ex ſingulis punctis P, B, p arcus elliptici PA, BA, pA in directum extendantur, itaut quælibet PQ ſit rectæ CA parallela & quadranti elliptico PTA æqualis: quod ſi ubique fieri concipiatur, puncta hæc Q ſita erunt in linea quadam curva AQDp, cujus naturam inveſtigare conſtitui.*

Ad geneſin hujus curvæ vel leviter attendenti mox patebit, eam ſequentes habere proprietates, quas evolvam, antequam in ipſam hujus curvæ indolem diligentius inquiram; ut ejus figura & ductus ſaltem obiter perſpiciatur.

§. IV. Primum igitur ſi in recta indefinita CB*p* quæ ad datam CA eſt normalis, capiatur quævis abſciſſa CP, applicata PC, quæ ei reſpondet, erit æqualis quadranti perimetri ellipſis, cujus ſemiaxes conjugati ſunt, recta data CA & ipſa abſciſſa CP. Hinc ſi capiatur abſciſſa CB = CA, quo caſu quadrans ellipticus abibit in quadrantem circularem AB, applicata reſpondens BD æqualis erit quartæ parti peripheriæ circuli radio AC deſcripti. Unde ſi ratio diametri ad peripheriam ponatur = 1 : π, erit iſta applicata

BD

BD $= \frac{1}{2} \pi$. AC: ſive ob $\pi = 3,14159265358979 32$ erit BD $= 1,5707963267948961$. AC.

§. V. Secundo: Si abſciſſa CP evaneſcat, ellipſis evadet infinite anguſta, atque cum linea recta confundetur. Hoc ergo caſu quadrans ellipticus abibit in ipſam lineam AC, cui propterea applicata abſciſſæ evaneſcenti reſpondens erit æqualis. Quare ipſa recta CA erit applicata puncto C reſpondens, & curva quæſita per punctum A tranſibit. Hujus ergo curvæ jam duo habemus puncta cognita A & D, quorum alterum A geometrice datur, alterum vero D per rationem diametri ad peripheriam definitur.

§. VI. Tertio: Ex cognito quovis curvæ puncto Q intra A & D ſito, ſemper aliud quoddam curvæ punctum q ultra D ſitum definiri poteſt. Capiatur enim Cp tertia proportionalis ad CP & CA, ut ſit $Cp = \frac{CA . CA}{CP.}$, quia eſt CP : CA = CA : Cp, erit quadrans ellipticus Ap ſimilis quadranti elliptico AP, cum utrinque eadem ſit ratio inter ſemiaxes conjugatos. Hinc erit arcus Ap ad arcum AP ut AC ad CP, ideoque pq : PQ = AC : CP ſeu $pq = \frac{AC.PQ}{CP}$. Conſequenter ſi curvæ quæſitæ arcus AD tantum jam fuerit deſcriptus, ex eo reliqua curvæ pars Dq in infinitum extenſa definietur.

§. VII. Quarto: Hinc jam inſignis proprietas æquationis, qua natura curvæ AQDq exprimitur, agnoſcitur. Si enim recta data AC unitate deſignetur, ut ſit AC $= 1$; abſciſſa autem quævis unitate minor CP $= p$, eique reſpondens applicata PQ $= q$; tum vero ponatur abſciſſa illa altera Cp = P & applicata pq = Q; erit $P = \frac{1}{p}$ & $Q = \frac{q}{p}$. Quare cum inter P & Q eadem

Q 2 eſſe

eſſe debeat æquatio quæ eſt inter p & q, patet æquationem inter p & q nullam mutationem eſſe ſubituram, ſi in ea loco p ubique ſcribatur $\frac{1}{p}$ & $\frac{q}{p}$ loco q. Unde qualis ipſius p functio ſit q conjicere licet.

§. VIII. Quinto: Patet creſcentibus abſciſſis CP applicatas continuo creſcere, cum ſemper ſint majores quam abſciſſæ. Verum ſi abſciſſæ ſtatuantur infinitæ, applicatæ ipſis fient æquales: diſcrimen enim prodibit infinite parvum; unde colligimus quæſitam curvam habere aſymtotam, & quidem rectam CV angulum rectum ACB biſecantem. Forma igitur hujus curvæ ſimilis erit hyperbolæ æquilateræ centrum in C, axem CA & aſymtotam CV habentis. Ex deſcriptione porro intelligitur, curvam infra rectam CA productam ſui ſimilem fore, ideoque rectam CA ejus fore diametrum perinde atque hyperbolæ. Veruntamen hoc facile perſpicitur, noſtram curvam multo lentius ad aſymtotam ſuam CV appropinquare quam hyperbolam. Nam in hyperbola æquilatera, cui noſtram curvam comparamus, quævis applicata PQ æqualis eſt rectæ lineæ AP; unde cum applicata noſtræ curvæ arcui AP ſit æqualis, patet hyperbolam noſtræ curvæ fore circum ſcriptam, ita tamen ut in initio A, & in ſpatio infinito ſe mutuo tangant.

§. IX. His affectionibus latius patentibus in genere notatis, in ipſam hujus curvæ naturam accuratius inquiramus, ac propoſita quacunque abſciſſa CP $= p$, valorem reſpondentis applicatæ PQ $= q$ inveſtigemus; qui cum expreſſione finita contineri nequeat, per ſeriem infinitam exhiberi debebit. Sequens igitur reſolvi oportet

Pro-

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Problema.

Ex datis ſemiaxibus CA & CP quadrantis elliptici CAP per ſeriem infinitam definire longitudinem arcus quadrantis ATP.

Solutio.

§. X. Cum vocatus ſit alter ſemiaxis $AC = 1$, alter vero $CP = p$, & arcus $AYP = q$, quæratur primo arcus quivis indefinitus PY, qui vocetur $= s$. Jam ducta ad CP applicata normali YX, ſit $CX = x$ & $XY = y$, erit ex natura ellipſis $x = p\sqrt{(1-yy)}$, hincque $dx = \frac{-pydy}{\sqrt{(1-yy)}}$. Fiet ergo ob $ds = \sqrt{(dx^2+dy^2)}$

$$ds = \frac{dy\sqrt{(1-yy+ppyy)}}{\sqrt{(1-yy)}}$$

unde integrando erit arcus $s = \int\frac{dy\sqrt{(1-yy+ppyy)}}{\sqrt{(1-yy)}}$

quæ integratio ita inſtitui debet, ut poſito $y = 0$ fiat quoque $s = 0$, quia evaneſcente applicata $XY = y$ ſimul $PY = s$ evaneſcit. Hoc igitur integrali invento ſi ponatur $y = CA = 1$, arcus indefinitus PY abibit in longitudinem quadrantis elliptici $PYA = q$, quem quærimus, ita ut ſit

$$q = \int\frac{dy\sqrt{(1-yy+ppyy)}}{\sqrt{(1-yy)}}$$

ſiquidem peracta integratione ponatur $y = 1$.

§. XI. Ad inſtitutum ergo noſtrum non eſt neceſſe, ut quæramus valorem integralis hujus indefiniti, ſed eum tantum, quem induit, ſi poſt integrationem variabili y tribuatur valor determinatus $= 1$: quo pacto ſeries multo ſimplicior valorem q exprimens obtineri poterit. Ponatur enim brevitatis gratia

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$1 - pp = nn$, ut ſit $\sqrt{(1 - yy + ppyy)} = \sqrt{(1 - nnyy)}$ eritque hanc formulam in ſeriem evolvendo.

$$\sqrt{(1 - nnyy)} = 1 - \tfrac{1}{2} nnyy - \frac{1.1}{2.4} n^4 y^4 - \frac{1.1.3}{2.4.6} n^6 y^6 - \&c.$$

Quo valore ſubſtituto pro $\sqrt{(1 - yy + ppyy)}$, arcus q ita exprimetur ut ſit:

$$q = \int \frac{dy}{\sqrt{(1 - yy)}} - \tfrac{1}{2} nn \int \frac{yydy}{\sqrt{(1 - yy)}} - \frac{1.1}{2.4} n^4 \int \frac{y^4 dy}{\sqrt{(1 - yy)}}$$
$$- \frac{1.1.3}{2.1.6} \int \frac{y^6 dy}{\sqrt{(1 - yy)}} \ \&c.$$

ſi quidem in ſingulis his integralibus poſt integrationem ponatur $y = 1$.

§. XII. Evolvamus ergo ſingula hæc integralia; ac primo quidem ex circulo manifeſtum eſt, formulam $\int \frac{dy}{\sqrt{(1 - yy)}}$ exprimere arcum circuli cujus ſinus $= y$ pro radio $= 1$: unde poſito $y = 1$, hæc formula dabit quartam peripheriæ partem, cujus radius $= 1$. Ideoque poſita ratione diametri ad peripheriam $= 1 : \pi$, erit $\int \frac{dy}{\sqrt{(1 - yy)}} = \frac{\pi}{2}$; ſicque jam adepti ſumus valorem primi termini in ſerie noſtra ante inventa.

§. XIII. Reliqui termini pari modo per valorem π commode poterunt exprimi; cujusvis enim termini integratio ad integrationem præcedentis reducitur: quod quo facilius intelligatur conſideremus formulam quamcunque $\int \frac{y^{\mu} dy}{\sqrt{(1 - yy)}}$; erit ſequens

ſy

$\int \frac{y^{\mu+2}\,dy}{\sqrt{(1-yy)}}$. Jam aſſumamus hanc formulam algebraicam $y^{\mu+1}\sqrt{(1-yy)}$, cujus differentiale cum ſit $=$

$$\frac{(\mu+1)y^{\mu}\,dy-(\mu+2)y^{\mu+2}\,dy}{\sqrt{(1-yy)}}$$ erit viciſſim

$$(\mu+1)\int\frac{y^{\mu}\,dy}{\sqrt{(1-yy)}}-(\mu+2)\int\frac{y^{\mu+2}\,dy}{\sqrt{(1-yy)}}=y^{\mu+1}\sqrt{(1-yy)}$$

unde colligimus fore

$$\int\frac{y^{\mu+2}\,dy}{\sqrt{(1-yy)}}=\frac{\mu+1}{\mu+2}\int\frac{y^{\mu}\,dy}{\sqrt{(1-yy)}}-\frac{1}{\mu+2}y^{\mu+1}\sqrt{(1-yy)}$$

Quare invento integrali $\int\frac{y^{\mu}\,dy}{\sqrt{(-yy)}}$ ex eo facile elicitur integrale ſequens $\int\frac{y^{\mu+2}\,dy}{\sqrt{(1-yy)}}$

§. XIV. Quoniam vero eos tantum horum integralium valores deſideramus, qui prodeunt poſito $y=1$; hoc caſu quantitas algebraica $\frac{1}{\mu+1}y^{\mu+1}\sqrt{(1-yy)}$ evaneſcit, eritque generatim pro caſu $y=1$

$\int y$

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$$\int \frac{y^{\mu+2}\, dy}{\surd(1-yy)} = \frac{\mu+1}{\mu+2} \int \frac{y^{\mu}\, dy}{\surd(1-yy)}$$

ſubſtituamus jam pro μ ſucceſſive valores 0, 2, 4, 6, 8 &c. & quoniam vidimus eſſe $\int \frac{dy}{\surd(1-yy)} = \frac{\pi}{2}$, erit, ut ſequitur:

$$\text{ſi } \mu = 0; \int \frac{y^2\, dy}{\surd(1-yy)} = \frac{1}{2} \int \frac{dy}{\surd(1-yy)} = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\mu = 2; \int \frac{y^4\, dy}{\surd(1-yy)} = \frac{3}{4} \int \frac{y^2\, dy}{\surd(1-yy)} = \frac{1.3}{2.4} \cdot \frac{\pi}{2}$$

$$\mu = 4; \int \frac{y^6\, dy}{\surd(1-yy)} = \frac{5}{6} \int \frac{y^4\, dy}{\surd(1-yy)} = \frac{1.3.5}{2.4.6} \cdot \frac{\pi}{2}$$

$$\mu = 6; \int \frac{y^8\, dy}{\surd(1-yy)} = \frac{7}{8} \int \frac{y^6\, dy}{\surd(1-yy)} = \frac{1.3.5.7}{2.4.6.8} \cdot \frac{\pi}{2}$$

unde lex, qua ſequentes progrediuntur, ſponte elucet.

§. XV. Quodſi jam iſti valores pro formulis integralibus, ex quibus longitudo quadrantis elliptici q conflari inventus eſt, ſubſtituantur, reperietur

$$q = \frac{\pi}{2} - \frac{1}{2} nn \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1.1}{2.4} n^4 \cdot \frac{1.3}{2.4} \cdot \frac{\pi}{2} - \frac{1.1.3}{2.4.6} n^7 \cdot \frac{1.3.5}{2.4.6} \cdot \frac{\pi}{2} \text{ \&c.}$$

quæ ad ſequentem ſeriem ſatis concinnam revocatur

$$q = \frac{\pi}{2}\left(1 - \frac{1.1}{2.2} n^2 - \frac{1.1.1.3}{2.2.4.4} n^4 - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} n^6 - \text{\&c.}\right)$$

cujus

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A.. x . x
B.. y y
C... z $x+y$
D... t $y-x$

$$z = x^2 + y^2$$

$$t = y^2 - x^2$$

$$(x^2+y^2)(x+y) = 715 \ldots (y^2-x^2)(y-x) = 99$$

$$(y-x)^2(y+x) = 99$$

$$(x^2+y^2)(x+y) = 715$$

$\frac{715}{23}$

$$\frac{x^2+y^2}{(y-x)^2} = \frac{715}{99} =$$

Van

A.. 4	16		
B.. 7	49	49	
	65	16	
C.. 65	65	33	
	715	3	
D.. 33		99	

A... $x-y$ ... $x-y$ / $x^2-2xy+y^2$
B... $x+y$ $x+y$ / $x^2+2xy+y^2$
$2x^2 + 2y^2$

C.. $2x^2+2y^2$ $2x$

D.. $4xy$ $2y$

$$\frac{2\left(30\frac{1}{4}\right)}{2\frac{1}{4}}$$

$$2\left(32\tfrac{1}{2}\right) = 65$$

$$4\left(5\tfrac{1}{2}\right)\left(1\tfrac{1}{2}\right) = 4\,\frac{11.3}{4} = 33$$

$$4x^3 + 4xy^2 = 715 \qquad 8xy^2 = 99$$

$$8x^3 + 8xy^2 = 1430 \qquad x = 5\tfrac{1}{2}$$

$$8x^3 = 1331 \qquad 8y^2 = 18$$

$$2x = 11 \qquad 4yy = 9$$

$$x = 5\tfrac{1}{2} \qquad 2y = 3$$

$$y = 1\tfrac{1}{2}$$

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$$\int \frac{y^{\mu+2}\, dy}{\sqrt{(1-yy)}} = \frac{\mu+1}{\mu+2} \int \frac{y^{\mu}\, dy}{\sqrt{(1-yy)}}$$

ſubſtituamus jam pro μ ſucceſſive valores 0, 2, 4, 6, 8 &c. & quoniam vidimus eſſe $\int \frac{dy}{\sqrt{(1-yy)}} = \frac{\pi}{2}$, erit, ut ſequitur:

$$\text{ſi } \mu = 0;\ \int \frac{y^2\, dy}{\sqrt{(1-yy)}} = \frac{1}{2} \int \frac{dy}{\sqrt{(1-yy)}} = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\mu = 2;\ \int \frac{y^4\, dy}{\sqrt{(1-yy)}} = \frac{3}{4} \int \frac{y^2\, dy}{\sqrt{(1-yy)}} = \frac{1.\,3}{2.\,4} \cdot \frac{\pi}{2}$$

$$\mu = 4;\ \int \frac{y^6\, dy}{\sqrt{(1-yy)}} = \frac{5}{6} \int \frac{y^4\, dy}{\sqrt{(1-yy)}} = \frac{1.\,3.\,5}{2.\,4.\,6} \cdot \frac{\pi}{2}$$

$$\mu = 6;\ \int \frac{y^8\, dy}{\sqrt{(1-yy)}} = \frac{7}{8} \int \frac{y^6\, dy}{\sqrt{(1-yy)}} = \frac{1.\,3.\,5.\,7}{2.\,4.\,6.\,8} \cdot \frac{\pi}{2}$$

unde lex, qua ſequentes progrediuntur, ſponte elucet.

§. XV. Quodſi jam iſti valores pro formulis integralibus, ex quibus longitudo quadrantis elliptici q conflari inventus eſt, ſubſtituantur, reperietur

$$q = \frac{\pi}{2} - \frac{1}{2} nn.\ \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1.\,1}{2.\,4} n^4.\ \frac{1.\,3}{2.\,4} \frac{\pi}{2} - \frac{1.\,1.\,3}{2.\,4.\,6} n^6.\ \frac{1.\,3.\,5}{2.\,4.\,6} \cdot \frac{\pi}{2} \text{ \&c.}$$

quæ ad ſequentem ſeriem ſatis concinnam revocatur

$$q = \frac{\pi}{2}\left(1 - \frac{1.\,1}{2.\,2} n^2 - \frac{1.\,1.\,1.\,3}{2.\,2.\,4.\,4} n^4 - \frac{1.\,1.\,1.\,3.\,3.\,5}{2.\,2.\,4.\,4.\,6.\,6} n^6 - \text{\&c.}\right)$$

cujus

cujus lex progreſſionis eſt manifeſta. Reſtituatur ergo pro nn ſuus valor $1 - pp$, eritque

$$q = \frac{\pi}{2}\left(1 - \frac{1.1}{2.2}(1-pp) - \frac{1.1.1.3}{2.2.4.4}(1-pp)^2 - \frac{1.1.1.3.3.5}{2.2.4.4.6.6}(1-pp)^3 - \&c.\right)$$

§. XVI. Cum pro curva noſtra AQDq littera p exhibeat abſciſſam CP & littera q applicatam PQ; jam adepti ſumus pro iſta curva æquationem inter ejus coordinatas p & q, quæ etſi ſerie conſtat infinita, tamen non ſolum ejus naturam in ſe complectitur, ſed etiam valores applicatæ q mox ſatis accurate exhibet, ſi abſciſſa p parum ab unitate differat: hoc eſt, cum ſit CB $=$ CA $= 1$, ſi punctum P ipſi B fuerit proximum; tum enim ob $1 - pp = nn$ quantitatem valde parvam ſeries inventa valde convergit.

§. XVII. Hinc igitur indolem noſtræ curvæ prope punctum D, hoc eſt ejus directionem & curvaturam definire poterimus. Primo enim patet uti jam vidimus, ſi $p = 1$ fore $q = \frac{\pi}{2}$, ita ut ſumta abſciſſa CB $= 1$ ſit applicata BD $= \frac{\pi}{2} = 1,5707963267948961$. Deinde ad poſitionem tangentis inveniendam, quæratur ratio differentialium $dq : dp$, quæ per differentiationem reperitur:

$$\frac{dq}{dp} = \frac{\pi}{2}p\left(\frac{1}{2} + \frac{1.1.3}{2.2.4}(1-pp) + \frac{1.1.3.3.5}{2.2.4.4.6}(1-pp)^2 + \frac{1.1.3.3.5.5.7}{2.2.4.4.6.6.8}(1-pp)^3 + \&c.\right)$$

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Poſito jam $p = 1$ fiet $\frac{dq}{dp} = \frac{\pi}{4}$. Unde ſi DG ſit tangens curvæ in puncto D, cum ſit BD : BG $= dq : dp$, erit BG $= \frac{dp}{dq}$. BD $= \frac{4}{\pi}$. BD & ob BD $= \frac{\pi}{2}$, fiet BH $= 2 = 2$BC, & CG $=$ BC. Sicque hoc caſu ſubtangens BG erit dupla abſciſſæ BC: & cum anguli BGD tangens ſit $= \frac{dq}{dp} = \frac{\pi}{4} = 0,78539816$ erit angulus BGD $= 38^{\circ}, 8^{I}, 45^{II}, 41^{III}, 51^{IV}$.

§. XVIII. Ad radium oſculi ſeu evolutæ in puncto D definiendum, cum ſit ob $\frac{dq}{dp} = \frac{\pi}{4}$, elementum curvæ $\sqrt{(dp^2 + dq^2)} = dp\sqrt{(1 + \frac{\pi\pi}{16})}$, erit radius oſculi $= (1 + \frac{\pi\pi}{16})^{3:2} dp^2 : ddq$.

At ſumendis differentialibus ſecundis erit

$$\frac{ddq}{dp^2} = \frac{\pi}{2}\left(\frac{1}{2} + \frac{1.1.3}{2.2.4}(1-pp) + \frac{1.1.3.3.5}{2.2.4.4.6}(1-pp)^2 + \&c\right)$$
$$-\frac{\pi}{2}pp\left(\frac{1.3}{2.4} + \frac{1.1.3.5}{2.2.4.6}(1-pp) + \frac{1.3.3.5.5.7}{2.2.4.4.6.7}(1-pp)^2 + \&c.\right)$$

Poſito ergo $p = 1$, erit $\frac{ddq}{dp^2} = \frac{\pi}{2}\left(\frac{1}{2} - \frac{3}{8}\right) = \frac{\pi}{16}$: Unde in puncto curvæ D erit radius evolutæ $= \frac{16}{\pi}(1 + \frac{\pi\pi}{16})\sqrt{(1 + \frac{\pi\pi}{16})}$, qui valor in numeris proxime reperitur $= 10,470672$.

§. XIX.

§. XIX. Poteſt hinc adhuc alia ſeries inveniri, quæ valorem applicatæ $PQ = q$ exprimat. Conſideretur enim illud alterum curvæ punctum q, pro quo ſit abſciſſa $Cp = P$ & applicata $pq = Q$, erit quoque ob $P > 1$.

$$Q = \frac{\pi}{2}\left(1 + \frac{1.1}{2.2}(PP - 1) - \frac{1.1.1.3}{2.2.4.4}(PP - 1)^2 + \frac{1.1.1.3.3.5}{2.2.4.4.6.6}(PP - 1)^3 - \&c.\right)$$

Jam vero ſupra notavimus, ſi ſit $P = \frac{1}{p}$, fore $Q = \frac{q}{p}$: quare his valoribus ſubſtitutis impetrabimus novam æquationem inter p & q, qua natura curvæ pariter exprimetur.

$$q = \frac{\pi}{2}\, p\left(1 + \frac{1.1}{2.2}\frac{(1 - pp)}{pp} - \frac{1.1.1.3}{2.2.4.4}\frac{(1 - pp)^2}{p^4} + \frac{1.1.1\,3.3.5\,(1 - pp)^3}{2.2.4.4.6.6.\;\; p^6} - \&c.\right)$$

quæ ſi cum ante inventa combinetur, innumerabiles aliæ novæ æquationes obtineri poterunt. Veluti ſi prior per p multiplicata ab hac ſubtrahatur, prodibit.

$$q - pq = \frac{\pi}{2}\, p\left(\frac{1.1}{2.2}\frac{(1 - pp)(1 + pp)}{pp} - \frac{1.1.1.3}{2.2.4.4}\frac{(1 - pp)^2(1 - p^4)}{p^4} + \&c.\right)$$

quæ reducitur ad hanc:

$$q = \frac{\pi}{4}(1+p)\left(\frac{1}{2}\cdot\frac{1+pp}{p} - \frac{1\cdot1\cdot3}{2\cdot4\cdot4}\frac{(1-p^4)(1-pp)}{p^3} + \frac{1\cdot1\cdot3\cdot3\cdot5}{2\cdot4\cdot4\cdot6\cdot6}\frac{(1+p^6)(1-pp)^2}{p^5} - \&c.\right)$$

vel cum ſeries adhuc ſit diviſibilis per $\frac{1+pp}{2p}$ erit

$$q = \frac{\pi}{8}\frac{(1+p)(1+pp)}{p}\left(1 - \frac{1\cdot3}{4\cdot4}\frac{(1-pp)}{pp}(1-pp) + \frac{1\cdot3\cdot3\cdot5}{4\cdot4\cdot6\cdot6}\frac{(1-pp+p^4)}{p^4}(1-pp)^2 - \frac{1\cdot3\cdot3\cdot5\cdot5\cdot7}{4\cdot4\cdot6\cdot6\cdot8\cdot8}\frac{(1-pp+p^4-p^6)}{p^6}(1-pp)^3 + \&c.\right)$$

§. XX. Manifeſtum autem eſt has ſeries parum ſubſidii afferre, ſi applicatas invenire velimus, quæ longius a BD, quæ abſciſſæ $p = 1$ reſpondet, ſint remotæ, ſi enim pro p ponatur numerus vel valde magnus vel valde parvus, ſeries inventa vel parum admodum convergit vel etiam divergit. Si enim inde longitudinem primæ applicatæ CA, quæ abciſſæ $p = 0$ reſpondet, definire velimus, ſerie primum inventa uti conveniet, quia in reliquis termini evadunt infinite magni. Habebimus igitur pro hoc caſu $p = 0$;

$$q = \frac{\pi}{2}\left(1 - \frac{1\cdot1}{2\cdot2} - \frac{1\cdot1\cdot1\cdot2}{2\cdot2\cdot4\cdot4} - \frac{1\cdot1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6\cdot6} - \frac{1\cdot1\cdot1\cdot3\cdot3\cdot5\cdot5\cdot7}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8} - \&c.\right)$$

quæ tam lente convergit, ut etiamſi plurimi termini actu colligerentur, tamen verus ipſius q valor, quem novimus eſſe $= 1$, inde difficillime agnoſci poſſet.

§. XXI.

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§. XXI. Quanquam autem nunc quidem novimus eſſe

$$1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \&c. = \frac{2}{\pi}$$

tamen inventio ſummæ hujus ſeriei non parum ardua videtur, ſi a priori tentetur. Veritatem quidem ex formula, quam quondam Walliſius pro circuli quadratura dedit, intelligere licet, ſi termini ab initio in unum colligantur, ſic enim prodit

$$1 - \frac{1.1}{2.2} = \frac{1.3}{2.2}$$

$$\frac{1.3}{2.2} - \frac{1.1.1.3}{2.2.4.4} = \frac{1.3.(4.4 - 1.1)}{2.2.\ 4.4} = \frac{1.3.3.5}{2.2.4.4}$$

$$\frac{1.3.3.5}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} = \frac{1.3.3.5.5.7}{2.2.4.4.6.6}$$

unde valor ſeriei in infinitum continuatæ erit

$$\frac{1.3.3.5.5.7.7.9.9.11.11.13.13}{2.2.4.4.6.6.8.8.10.10.12.12.14}\ \&c.$$

quæ expreſſio cum ſit ipſa Walliſiana, patet ſummam noſtræ ſeriei eſſe $= \frac{2}{\pi}$. Interim tamen juvabit tradere methodum hanc ſeriem aliasque ſimiles a priori ſummandi.

Problema.

Invenire ſummam hujus ſeriei infinitæ:

$$1 - \frac{1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \&c.$$

cujus lex progreſſionis primo intuitu eſt manifeſta.

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Solutio.

§. XXII. Ponatur ſumma hujus ſeriei, quæ quæritur $= s$, ut ſit

$$s = 1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \&c.$$

Jam eligatur ſeries, cujus ſumma conſtat, & cujus coefficientes jam in his terminis contineantur. Cujusmodi eſt hæc

$$\frac{1}{\surd(1-xx)} = 1 + \tfrac{1}{2}xx + \frac{1.3.}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \frac{1.3.5.7}{2.4.6.8}x^8 + \&c.$$

Erit ergo per differentiale quodpiam dP multiplicando & integrando

$$\int\frac{dP}{\surd(1-xx)} = P + \tfrac{1}{2}\int xx\,dP + \frac{1.3}{2.4}\int x^4\,dP + \frac{1.3.5}{2.4.6}\int x^6\,dP + \&c.$$

Nunc differentiale hoc dP ita definiatur, ut ſi poſt integrationem ponatur $x = 1$. fiat

$$\int x\,dP = -\tfrac{1}{2}P$$

$$\int x^4\,dP = +\tfrac{1}{2}\int xx\,dP = -\frac{1.1}{2.4}P$$

$$\int x^6\,dP = +\tfrac{3}{6}\int x^4\,dP = -\frac{1.1.3}{2.4.6}P$$

$$\int x^8\,dP = +\tfrac{5}{8}\int x^6\,dP = -\frac{1.1.3.5}{2.4.6.8}P$$

quo facto ſi hi valores ſubſtituantur habebitur;

$$\int\frac{dP}{\surd(1-xx)} = P\left(1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \&c.\right)$$

ideoque

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ideoque $\int \frac{dP}{\sqrt{(1-xx)}} = Ps$, ſi quidem poſt integrationem ſtatuatur $x = 1$.

§. XXIII. Huc ergo res redit, ut quæratur formula differentialis dP, ut ſuperioribus conditionibus ſatisfiat.

ſeu ut in genere ſit $\int x^{\mu+2}\,dP = \frac{\mu-1}{\mu-2}\int x^{\mu}\,dP$, ſi quidem poſt integrationem utramque ponatur $x = 1$. Omiſſa igitur hac conditione ſit

$$\int x^{\mu+2}\,dP = \frac{\mu-1}{\mu+2}\int x^{\mu}\,dP + \frac{Qx^{\mu+2}}{\mu+2}$$

ubi Q ejus modi ſit functio ipſius x, quæ evaneſcat poſito $x = 1$.

Capiantur ergo differentialia, eritque per x^{μ} dividendo

$$xxdP = \frac{\mu-1}{\mu+2}\,dP + \frac{xdQ + (\mu+1)\,Qdx}{\mu+2}$$

ſeu $0 = (\mu-1)\,dP - (\mu+2)\,xxdP + xdQ + (\mu+1)\,Qdx$ quæ æquatio, cum locum habere debeat pro omni valore ipſius μ, reſolvetur in has duas:

$$0 = dP - xxdP + Qdx$$
$$0 = -dP - 2xxdP + xdQ + Qdx$$

unde fit $dP = \frac{-Qdx}{1-xx} = \frac{xdQ + Qdx}{1+2xx}$

& $xdQ\,(1-xx) = -Qdx\,(2+xx)$

Quæ

Quare cum ſit $\frac{dQ}{Q} = \frac{dx(2+xx)}{x(1-xx)} = -\frac{2dx}{x} - \frac{3xdx}{1-xx}$

erit $Q = \frac{-(1-xx)^{\frac{3}{2}}}{xx}$ & $dP = \frac{dx}{xx}\sqrt{(1-xx)}$

§. XXIV. Verum hic notandum eſt, etſi valor ipſius Q evaneſcat poſito $x=1$; tamen caſu $\mu = 0$, quantitatem algebraicam $\frac{Qx^{\mu+1}}{\mu+2}$ non evaneſcere, ſi ponatur $x=0$, quæ tamen conditio æque eſt neceſſaria atque altera, ita ut hoc caſu non ſit $\int xxdP = -\frac{1}{2}P$. Cum autem reliquæ formulæ quibus $\mu > 0$ locum habeant, a formula $\int xxdP$ erit incipiendum, eritque

$$\int x^4 dP = \frac{1}{4}\int xxdP$$

$$\int x^6 dP = \frac{3}{6}\int x^4 dP = \frac{1.3}{4.6}\int xxdP$$

$$\int x^8 dP = \frac{5}{8}\int x^6 dP = \frac{1.3.5}{4.6.8}\int xxdP$$

&c.

unde habebitur

$$\int\frac{dP}{\sqrt{(1-xx)}} = P + \int xxdP\left(\frac{1}{2} + \frac{1.1.3}{2.4.4} + \frac{1.1.3.3.5}{2.4.4.6.6} + \&c.\right)$$

At eſt $\frac{1}{2} + \frac{1.1.3}{2.4.4} + \frac{1.1.3.3.5}{2.4.4.6.6} + \&c. = 2(1-s)$; ideoque

$\int dP$

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$\int \frac{dP}{\sqrt{(1-xx)}} = P + 2(1-s)\int xxdP$. At ob $dP = \frac{dx}{xx}\sqrt{(1-xx)}$

erit $P = C - \frac{\sqrt{(1-xx)}}{x} - A \text{ ſin } x$;

$$\int xxdP = \int dx \sqrt{(1-xx)} =$$

$\frac{1}{2} A \text{ ſin } x + \frac{1}{2} x \sqrt{(1-xx)}$, & $\int \frac{dP}{\sqrt{(1-xx)}} = D - \frac{1}{x}$ ubi conſtantes C & D ita accipi debent, ut integralia hæc evaneſcant poſito $x = 0$: quanquam autem utraque ſeorſim ſit infinita, tamen conjunctæ ſe mutuo deſtruent. Erit enim

$$\int \frac{dP}{\sqrt{(1-xx)}} - P = D - \frac{1}{x} - C + \frac{\sqrt{(1-xx)}}{x} + A \text{ ſin } x$$

quæ ut evaneſcat poſito $x = 0$, debet eſſe $D = C$, ideoque poſito jam $x = 1$ fiet $\int \frac{dP}{\sqrt{(1-xx)}} - P = -1 + \frac{\pi}{2}$: & quia eodem hoc caſu eſt $\int xxdP = \frac{\pi}{4}$, prodibit

$$-1 + \frac{\pi}{2} = 2(1-s)\frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{2}s,$$

hincque colligitur fore $\frac{\pi}{2}s = 1$ & $s = \frac{2}{\pi}$ ſeu

$$1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \& = \frac{2}{\pi}$$

uti ex rei natura jam concluſimus.

 §. XXX.

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§. XXV. Quoniam igitur eruimus in ipſo initio eſſe applicatam curvæ CA $= 1$, indolem hujus curvæ prope punctum A indagemus, ſeu in valorem applicatæ q inquiramus, ſi abſciſſa p fuerit valde parva. In hunc finem ponamus iterum $1 - pp = nn$, & cum ſit

$$q = \frac{\pi}{2}\left(1 - \frac{1.1}{2.2}nn - \frac{1.1.1.3}{2.2.4.4}n^4 - \frac{1.1.1.3.3.5}{2.2.4.4.6.6}n^6 - \&c.\right)$$

& quia novimus fore proxime $q = 1$, addamus æqualitatem modo inventam:

$$0 = 1 - \frac{\pi}{2}\left(1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6}n^6 - \&c.\right)$$

atque habebimus:

$$q = 1 + \frac{\pi}{2}\left(\frac{1.1}{2.2}(1 - nn) + \frac{1.1.1.3}{2.2.4.4}(- n^4) + \frac{1.1.1.3.3.5}{2.2.4.4.6.6}(1 - n^6) + \&c.\right)$$

cujus ſeriei cum ſinguli temini ſint per $1 - nn = pp$ diviſibiles, reducetur hæc expreſſio ad hanc:

$$q = 1 + \frac{\pi}{8}pp\left(1 + \frac{1.3}{4.4}(1 + nn) + \frac{1.3.3.5}{4.4.6.6}(1 + nn + n^4) + \frac{1.1.3.3.5.5.7}{4.4.6.6.8.8}(1 + n^2 + n^4 + n^6) + \&c.\right)$$

§. XXVI. Quodſi in hac expreſſione ſinguli termini ad poteſtates ipſius n evolvantur, reperietur

+

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$$q = 1 + \frac{\pi}{2}pp\left\{\begin{array}{l}
+\frac{1.1}{2.2}+\frac{1.1.1.3}{2.2.4.4}+\frac{1.1.1.3.3.5}{2.2.4.4.6.6}+ \\
\frac{1.1.1.3.3.5.5.7}{2.2.4.4.6.6.8.8}+\&c. \\
+n^2\left(\frac{1.1.1.3}{2.2.4.4}+\frac{1.1.1.3.3.4}{2.2.4.4.6.6}+\right. \\
\frac{1.1.1.3.3.5.5.7}{2.2.4.4.6.6.8.8}+\&c. \\
+n^4\left(\frac{1.1.1.3.3.5}{2.2.4.4.6.6}+\frac{1.1.1.3.3.5.5.7}{2.2.4.4.6.6.8.8}+\right. \\
\qquad\&c. \\
+n^6\left(\frac{1.1.1.3.3.5.5.7}{2.2.4.4.6.6.8.8}+\&c.\right. \\
\qquad\&c.
\end{array}\right.$$

At ex ſupra inventis habemus ſummam primæ ſeriei

$$\frac{1.1}{2.2}+\frac{1.1.1.3}{2.2.4.4}+\frac{1.1.1.3.3.5}{2.2.4.4.6.6}+\&c. = 1 - \frac{2}{\pi}$$

quæ ſi primo termino multetur, prodibit ſecunda, quæ eſt coefficiens ipſius nn, ita ut ſit

$$\frac{1.1.1.3}{2.2.4.4}+\frac{1.1.1.3.3.5}{2.2.4.4.6.6}+\&c. = \frac{1.3}{2.2}-\frac{2}{\pi}$$

hæc denuo primo termino multata, dabit coefficientem ipſius n^4, nempe

$$\frac{1.1.1.3.3.5}{2.2.4.4.6.6}+\&c. = \frac{1.3.3.5}{2.2.4.4}-\frac{2}{\pi}$$

ſimilique modo coefficiens ipſius n^6 erit $= \frac{1.3.3.5.5.7}{2.2.4.4.6.6} - \frac{2}{\pi}$. & ita porro, ſicque tandem obtinebitur.

$$q = 1 + \frac{\pi}{2}\, pp \Big((1 - \frac{2}{\pi}) + (\frac{1.3}{2.2} - \frac{2}{\pi})\, nn + (\frac{1.3.3.5}{2.2.4.4} - \frac{2}{\pi})\, n^4 + (\frac{1.3.3.5.5.7}{2.2.4.4.6.6} - \frac{2}{\pi})\, n^6 + \&c. \Big)$$

vel erit;

$$q = 1 + pp\, ((\frac{\pi}{2} - 1) + (\frac{1.3}{2.2}\,\frac{\pi}{2} - 1)\, nn + (\frac{1.3.3.5.}{2.2.4.4}\,\frac{\pi}{2} - 1)\, n^4 + (\frac{1.3.3.5.5.7}{2.2.4.4.6.6}.\,\frac{\pi}{2} - 1)\, n^6 + \&c.)$$

§. XXVII. Ponamus jam hic $n = 1$, ut obtineamus æquationem hujus formæ $q = 1 + App$, qua natura curvæ prope punctum A exprimitur: cum enim conjectare liceat veram æquationem futuram eſſe hujus formæ:

$$q = 1 + App + Bp^4 + Cp^6 + Dp^8 + \&c.$$

ſi abſciſſa p valde parva aſſumatur, reliqui termini præter binos primos ommitti poterunt, atque ex æquatione $q = 1 + App$, tam poſitio tangentis, quam curvatura in puncto A colligi poterit. Poſito enim $AR = x$, $RQ = y$, erit $q = 1 + x$ & $p = y$, ideoque ſi arcus AQ fuerit minimus, is cum parabola confundetur, cujus æquatio $x = Ayy$ ſeu $yy = \frac{1}{A}x$, ac propterea $\frac{1}{A}$ parameter. Unde ſequitur tangentem curvæ in A fore ad rectam AC perpendicularem, & radium oſculi ibidem eſſe $= \frac{1}{2A}$.

§. XXVIII.

§. XXVIII. Hic igitur coefficiens A reperietur, ſi in ſuperiore ſerie, per quam quantitas pp multiplicatur, ponatur $n = 1$; ita ut ſit

$$A = \left(\frac{\pi}{2} - 1\right) + \left(\frac{1.2}{2.2} \cdot \frac{\pi}{2} - 1\right) + \left(\frac{1.3.3.5}{2.2.4.4} \cdot \frac{\pi}{2} - 1\right) + \&c.$$

quæ autem ſi ejus ſummatio tentetur, tam parum convergens deprehenditur, ut ejus ſummam adeo infinitam ſuſpicari debeamus. In hac autem ſuſpicione eo magis confirmamur, ſi ſeriem primo (§. 15.) inventam, ſecundum dimenſiones ipſius p evolvamus, unde ſit

$$= \frac{\pi}{2} \begin{cases} 1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \&c. \\ + pp\left(\frac{1.1}{2.2} + \frac{1.1.1.2}{2.2.4.4} 2 + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} . 3 + \&c.\right) \\ - p^4\left(\frac{1.1.1.3}{2.2.4.4} + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} . 3 + \&c.\right) \end{cases}$$

§. XXIX. Hinc ergo coefficiens ipſius pp in æquatione generali pro curva $q = 1 + App + Bp^4 + Cp^6 + Dp^8 + \&c.$ erit

$$A = \frac{\pi}{2} \left(\frac{1.1}{2.2} . 1 + \frac{1.1.1.3}{2.2.4.4} . 2 + \frac{1.1.1.3.3.6}{2.2.4.4.6.6} 3 + \&c.\right)$$

$$\text{ſeu } A = \frac{\pi}{4} \left(\frac{1}{2} + \frac{1.1.3}{2.2.4} + \frac{1.1.3.3.5}{2.2.4.4.6} + \frac{1.1.3.3.5.5.7}{2.2.4.4.6.6.8} + \&c.\right)$$

ſimilique modo & reliquos coefficientes B, C, D. &c. ex hac ſerie eruere licebit. Verum hoc labore ſuperſedere poterimus, cum liqueat non ſolum coefficientem A, ſed etiam omnes reliquos prodituros

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dituros eſſe infinitos. Perſpicuum hoc fiet ex ſolutione hujus problematis.

Problema.

Invenire ſummam hujus ſeriei infinitæ:

$$S = \frac{1}{2} + \frac{1.1.3}{2.2.4} + \frac{1.1.3.3.5}{2.2.4.4.6} + \&c.$$

Solutio.

§. XXX. Aſſumatur ad hanc ſummam s inveniendam hæc formula:

$$\frac{1}{\sqrt{(1-xx)}} = 1 + \tfrac{1}{2}xx + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \&c.$$

ut ſit

$$\int \frac{dP}{\sqrt{(1-xx)}} = P + \tfrac{1}{2}\int xx\,dP + \frac{1.3}{2.4}\int x^4 dP + \frac{1.3.5}{2.4.6}\int x^6 dP$$

$$+ \&c.$$

ſitque ſi poſt integrationes ſingulas ponatur $x = 1$

$$\int xx\,dP = \frac{3}{4}P$$

$$\int x^4 dP = \frac{5}{6}\int xx\,dP = \frac{3.5}{4.6}P$$

$$\int x^6 dP = \frac{5}{8}\int x^4 dP = \frac{3.5.7}{4.6.8}P$$

&c.

hincque fiet

$\int dP$

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$$\frac{dP}{\sqrt{(1-xx)}} = P\left(1 + \frac{1.3}{2.3} + \frac{1.3.3.5}{2.4.4.6} + \frac{1.3.3.5.5.7}{2.4.4.6.6.8} + \&c.\right)$$

ſive $\int \frac{dP}{\sqrt{(1-xx)}} = 2Ps$; unde invento P, reperietur s ſi poſt integtationem ponatur $x = 1$.

§. XXXI. Cum igitur generaliter eſſe debeat.

$$\int x^{\mu+1}\, dP = \frac{\mu+3}{\mu+4} \int x^{\mu}\, dP + \frac{x^{\mu+1}\, Q}{\mu+4}$$

dummodo Q ejusmodi ſit functio, quæ evaneſcat poſito $x = 1$, erit

$$(\mu+4)\, xxdP = (\mu+3)\, dP + xdQ + (\mu+1)\, Qdx$$

unde duæ ſequentes æquationes conficiuntur.

$$xxdP = dP + Qdx$$

$$4xxdP = 3\,dP + dQ + Qdx$$

$$\& \; dP = \frac{-Q\,dx}{1-xx} = \frac{-xdQ - Q\,dx}{3-4xx}$$

hincque elicitur $\frac{dQ}{Q} = \frac{2\,dx - 3\,xxdx}{x(1-xx)} = \frac{2\,dx}{x} \quad \frac{-xdx}{1-xx}$

& $Q = -xx\sqrt{(1-xx)}$. Quare habebitur

$$dP = \frac{xxdx}{\sqrt{(1-xx)}} \;\&\; \frac{dP}{\sqrt{(1-xx)}} = \frac{xxdx}{1-xx} = -dx + \frac{dx}{1-xx}.$$

Fiet ergo $P = \frac{1}{4}\pi$, ſi poſt integrationem ponatur $x = 1$

at

at $\int \frac{dP}{\sqrt{(1-xx)}} = -x + \frac{1}{2} l \frac{1+x}{1-x}$, cujus valor poſito $x = 1$ ſit utique infinitus. Erit igitur $s = \infty$, ſeu ſumma ſeriei propoſitæ infinite magna.

§. XXXII. Quia igitur coefficiens A ipſius pp, in æquatione

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + \&c.$$

eſt infinitus, radius oſculi curvæ inpuncto A utique erit infinite parvus. Verum præterea hæc æquatio: in qua omnes omnino coefficientes A, B, C, D &c. fiunt infiniti, nihil plane ad curvæ cognitionem confert. Quia enim radius oſculi curvæ in A eſt infinite parvus, natura curvæ circa punctum A hujus modi æquatione $q = 1 + \alpha p^m$ exprimetur, in qua exponens m binario ſit minor, verumtamen unitate major: ſed ex omnibus, quæ hactenus ſunt tradita nulla via patet, qua hunc exponentem m ſcrutari queamus. Cum enim is numerus integer eſſe nequeat, nulla ſerierum, quas pro q eruimus, ita eſt comparata, ut ex ea poteſtatem ipſius p irrationalem eiicere liceat.

§. XXXIII. Hinc intelligimus problema eſſe ſummopere difficile, quo æquatio tantum elementaris requiritur, quæ naturam curvæ propoſitæ AQDq ſaltem proxime circa punctum A exhibeat. Notum eſt enim ſi ponatur AR $= x$ & RQ $= y$, quæcunque fuerit curva AQ, naturam minimæ ejus portiunculæ circa A ſemper hujusmodi æquatione $y^m = Ax$, comprehendi poſſe; ſiquidem curva ſit algebraica; pro curvis autem tranſcendentibus certum videtur, quasvis earum minimas portiunculas cum arcubus curvarum algebraicarum comparari poſſe. Quare in noſtra curva, etſi

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etſi eſt tranſcendens, hoc eo magis mirum videri debet, quod nulla hujusmodi formula $y = Ax^m$ exhiberi poſſit, quæ ſaltem minimæ ejus portiunculæ circa A ſitæ naturam declaret.

§. XXXIV. Hunc nodum ut reſolvamus, æquationem nobis finitam inter coordinatas p & q inveſtigare oportebit, quæ etſi, ut facile prævidere licet, ad differentialia ſecundi ordinis exſurget, tamen ad accuratiorem curvæ cognitionem magis erit accomodata. Eliciemus autem hujusmodi æquationem, quæ numero terminorum finito conſtet, ſi ſeriem primo inventam ad ſummam revocabimus. Cum enim poſito $1 - pp = nn$ ſit

$$\frac{2q}{\pi} = 1 - \frac{1.1}{2.2}nn - \frac{1.1.1.3}{2.2.4.4}n^4 - \frac{1.1.1.3.3.5}{2.2.4.4.6.6}n^6 - \&c.$$

Erit differentiando

$$\frac{2dq}{\pi dn} = -\frac{1.1}{2}n - \frac{1.1.1.3}{2.2.4}n^3 - \frac{1.1.1.3.3.5}{2.2.4.4.6}n^5 - \&c.$$

quæ per n multiplicata denuoque differentiata dat

$$\frac{2}{\pi dn}\, d\,\frac{ndq}{dn} = -1.1\,n - \frac{1.1}{2.2}1.3\,n^3 - \frac{1.1.1.3}{2.2.4.4}3.5\,n^5 - \&c.$$

Multiplicetur hæc per $\frac{dn}{n}$ ac rurſus integretur, erit

$$\frac{2}{\pi}\int\frac{1}{n}\,d.\frac{ndq}{dn} = -1.n - \frac{1.1}{2.2}.1\,n^3 - \frac{1.1.1.3}{2.2.4.4}.\,3\,n^5 - \&c.$$

Multiplicetur per $\frac{dn}{n^3}$ & integrando prodibit,

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$$\frac{2}{\pi}\int\frac{dn}{n^3}\int\frac{1}{n}\,d.\frac{ndq}{dn}=\frac{1}{n}-\frac{1.1}{2.2}n-\frac{1.1.1.3}{2.2.4.4}n^3-\&c.$$

quæ ſeries cum ſit ipſa propoſita, per n diviſa erit

$$\frac{2}{\pi}\int\frac{dn}{n^3}\int\frac{1}{n}\,d.\frac{ndq}{dn}=\frac{2q}{\pi n}\text{ ſeu }\int\frac{dn}{n^3}\int\frac{1}{n}\,d.\frac{ndq}{dn}=\frac{q}{n}$$

§. XXXV. Sumamus nunc differentialia, habebiturque

$$\frac{dn}{n^3}\int\frac{1}{n}\,d.\frac{ndq}{dn}=\frac{ndq-qdn}{nn}\text{ ſeu}$$

$$\int\frac{1}{n}\,d.\frac{ndq}{dn}=\frac{nndq}{dn}-nq.$$

porroque differentiando

$$\frac{1}{n}\,d.\frac{ndq}{dn}=nd.\frac{ndq}{dn}+ndq-ndq-qdn$$

$$\text{ſeu }(1-nn)\,d.\frac{ndq}{dn}+qndn=0$$

Jam ob $1-nn=pp$ erit $ndn=-pdp$ & $\frac{dn}{n}=-\frac{pdp}{1-pp}$

unde fit $-ppd.\frac{(1-pp)\,dq}{pdp}-pqdp=0$ ſeu

$d.\frac{(1-pp)\,dq}{pdp}+\frac{qdp}{p}=0.$ Sumatur jam dp conſtans erit

$$\frac{(1-pp)\,dq}{pdp}-\frac{dq\,(1+pp)}{pp}+\frac{qdp}{p}=0\text{ ſeu}$$

$$p\,(1-pp)\,ddq-dpdq\,(1+pp)+pqdp^2=0.$$

§. XXXVI.

§. XXXVI. En igitur æquationem differentialem ſecundi gradus pro curva propoſita

$$p\,(1-pp)\,ddq - dpdq\,(1+pp) + pqdp^2 = 0$$

ex qua poteſtas illa ipſius p in æquatione $q = 1 + Ap^m$ elici debet, ſi abſciſſa p valde parva ſtatuatur. Cum igitur fiat $dq = m\,Ap^{m-1}\,dp$ & $ddq = m\,(m-1)\,Ap^{m-2}\,dp$ orietur

$$\left.\begin{array}{l} m\,(m-1)\,Ap^{m-1} - mAp^{m-1} + p \\ -m\,(m-1)\,Ap^{m+1} - m\,Ap^{m+1} + Ap^{m+1} \end{array}\right\} = 0.$$

ſeu $m\,(m-2)\,Ap^{m-1} - mmAp^{m+1} + p = 0.$

Deberet ergo eſſe $m = 2$, ut terminus Ap^{m-1} cum p comparari poſſet, ſed tum iterum obtinetur $A = \infty$: præterea vero hinc perſpicitur exponentem m nullo modo numerum fractum eſſe poſſe, ita ut hinc difficultas ſupra memorata augeri potius quam tolli videatur.

§. XXXVII. Quodſi regulis conſuetis uti velimus ad æquationem inventam in ſeriem evolvendam, quæ ſecundum poteſtates ipſius p procedat, quoniam novimus primum ſeriei terminum eſſe $= 1$, nullam aliam formam inde colligere licet niſi hanc:

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \&c.$$

unde fit

$$\frac{dq}{dp} = 2Ap + 4Bp^3 + 6Cp^5 + 8\,Dp^7 + \&c.$$

$$\& \quad \frac{ddq}{dp^2} = 2A + 12Bp^2 + 30Cp^4 + 56Dp^6 + \&c.$$

qui valores in æquatione ſubſtituti præbebunt:

$$\left.\begin{array}{l} +2Ap + 12Bp^3 + 30Cp^5 + 56Dp^7 + \&c. \\ \quad -2A - 12B - 30C - \&c. \\ -2Ap - 4B - 6C - 8D - \&c. \\ \quad -2A - 4B - 6C - \&c. \\ +p + A + B + C + \&c. \end{array}\right\} = 0$$

unde omnes coefficientes A, B, C, &c. prodeunt infiniti.

§. XXXVIII. Hinc igitur videmus regulas ordinarias, ſecundum quas vulgo forma ſeriei, in quam æquatio differentialis transmutanda ſit, dijudicari ſolet, non eſſe ſufficientes, cum hoc caſu nullam afferant utilitatem: unde noſtra æquatio eo majorem meretur attentionem. Sequenti tamen modo ex ea natura curvæ prope punctum A colligi poterit, ex quo ſimul intelligetur, quemadmodum quoque in aliis caſibus defectus iſte regularum uſu receptarum ſuppleri, eæque ad praxin accommodari debeant. Quia enim abſciſſam p hic pro infinite parva habemus, in æquatione pro $1 - pp$ & $1 + pp$ ponere licebit 1, & quia novimus eſſe hoc caſu proxime $q = 1$, pro quantitate finita q unitatem ſcribamus: quo facto æquatio differentio-differentialis inventa pro caſu, quo abſciſſa p eſt minima ſequentem induet formam;

$$pddq - dpdq + pdp^2 = 0.$$

§. XXXIX. Hujus jam æquationis reſolutio eſt facilis, cum enim dp ſit conſtans, ponatur $dq = rdp$, erit $ddq = drdp$, habebiturque:

$$pdr - rdp + pdp = 0$$

pdr

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$$\text{ſive}\quad \frac{pdr - rdp}{pp} + \frac{dp}{p} = 0$$

cujus integrale eſt: $\frac{r}{p} + lp = C$, unde fit

$$r = Cp - plp$$

ideoque $\quad dq = Cpdp - pdp\ lp$

Hæc jam æquatio integrata dabit:

$$q = 1 + \tfrac{1}{2} Cp^2 - pp\ lp + \tfrac{1}{4} pp$$

in qua cum terminus pp incomparabiliter ſit minor quam $pp\,ld$, erit pro curvæ initio A:

$$q = 1 - \tfrac{1}{2}\ pp\,lp$$

§. XL. Nunc igitur naturam curvæ prope initium A æquatione ſimplici definire poſſumus: ſi enim vocemus $AR = x$ & $RQ = y$, ob $p = y$ & $q = 1 + x$, orietur hæc $x = -\tfrac{1}{2}\ yy\,ly$, ad quam æquatio generalis pro curva revocatur, ſi coordinatæ x & y ſint quam minimæ. Patet igitur ne minimum quidem arculum circa A tanquam portiunculam curvæ algebraicæ ſpectari poſſe, ſed ejus naturam logarithmos implicare. Et quoniam æquatio logarithmica in exponentialem transformari poteſt, initium curvæ noſtræ A commune erit cum linea tranſcendente, cujus æquatio eſt $e^{-2x} = y^{yy}$, ſumto e pro numero cujus logarithmus hyperbolicus eſt $= 1$.

§. XLI. Æquatione hac $x = -\tfrac{1}{2} yy\,ly$ confirmantur quoque ea, quæ ſupra jam de affectionibus hujus curvæ in puncto A notavimus. Primo enim patet ſi ſit $y = 0$, fore quoque $yy\,ly$ ac proinde $x = 0$, etſi hoc caſu ſit $ly = -\infty$. Deinde cum ſit $dy = -ydy\,ly - \tfrac{1}{2} ydy$, quia y incomparabiliter eſt minus quam yly,

erit $dx = - ydy\,ly$, ac propterea $\frac{dy}{dx} = \frac{-1}{yly} = \infty$ poſito $y = 0$; unde patet tangentem curvæ in A ad abſciſſam A R eſſe perpendicularem. Porro cum ſit ſubnormalis $\frac{ydy}{dx} = \frac{-1}{ly}$, hocque caſu ſubnormalis radio evolutæ æquetur, ob $ly = \infty$ ſi $y = 0$, manifeſtum eſt radium oſculi curvæ in A eſſe infinite parvum.

§. XLII. Maxime autem differt hæc curva a curvis algebraicis, quæ in initio A quoque habent radium oſculi evaneſcentem. Curvarum enim algebraicarum, quæ hac indole gaudent, natura circa initium A hujusmodi formula exprimitur $x = \alpha y^m$ exiſtente $m < 2$, attamen $m > 1$. Sit igitur $m = 2 - n$ exiſtente n fractione unitate minore, ut ſit $x = \alpha y^{2-n}$, erit $dx = \alpha(2-n) y^{1-n} dy$, ideoque $\frac{dy}{dx} = \frac{1}{\alpha(1-n)y^{1-n}} = \infty$ ob $y^{1-n} = 0$: at radius oſculi, qui ſubnormali $\frac{ydy}{dx}$ æqualis eſt, erit $= \frac{y^n}{\alpha(2-n)}$ $= 0$. Pro noſtra vero curva radius oſculi inventus eſt $= \frac{-1}{ly}$, unde radius oſculi evaneſcens in curva algebraica quacunque erit ad radium oſculi in noſtræ curvæ puncto A ut $-y^n\,ly$ ad $\alpha(2-n)$ hoc eſt ut 0 ad 1; quantumvis enim exiguus ſit exponens n; caſu $y = 0$ ſemper eſt $y^n\,ly = 0$, etiamſi ſit $ly = -\infty$. Quare in noſtra quidem curva radius oſculi in A eſt infinite parvus, ſed tamen

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men infinities major eſt, quam radius oſculi evaneſcens in omni curva algebraica.

§. XLIII. Cognito jam initio ſeriei, qua valor applicatæ PQ $= q$ per abſciſſam CP $= p$ exprimitur, ſcilicet

$$q = 1 - \tfrac{1}{2} pplp + App$$

non difficile erit hinc formam totius ſeriei colligere. Cum enim ex æquatione differentio-differentiali intelligatur ſequentium terminorum poteſtates ipſius p binario creſcere, valor ipſius q generatim gemina ſerie infinita exprimetur, eritque

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \&c.$$
$$- \alpha pplp - \beta p^4 lp - \gamma p^6 lp - \delta p^8 lp - \&c.$$

in qua quidem nunc jam novimus eſſe $\alpha = \tfrac{1}{2}$.

§. XLIV. Cum igitur verus valor ipſius q duplici ſerie contineatur, ut utramque ſeorſim eliciamus, ponamus

$q = r - slp$ eritque differentiando

$$dq = dr - \frac{sdp}{p} - dslp$$

$$ddq = ddr - \frac{2dpds}{p} + \frac{sdp^2}{pp} - ddslp$$

Hi valores in noſtra æquatione differentiali

$$p\,(1 - pp)\,ddq - dpdq\,(1 + pp) + pqdp^2 = 0$$

ſubſtituantur, ac termini per lp affecti ſeorſim nihilo æquentur, hoc modo duæ obtinebuntur æquationes:

I. q

$$\text{I. } p\,(1-pp)\,dds - (1+pp)\,dpds + psdp^2 = 0$$

$$\text{II. } p\,(1-pp)\,ddr - (1+pp)\,dpdr + prdp^2 -$$

$$2\,(1-pp)\quad dpds + \frac{2sdp}{p} = 0$$

§. XLV. Ad has æquationes reſolvendas ponatur

$$r = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \&c.$$

$$s = \alpha p^2 + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p^{10} + \&c.$$

eritque differentialibus ſumendis

$$\frac{dr}{dp} = 2Ap + 4Bp^3 + 6Cp^5 + 8Dp^7 + \&c.$$

$$\frac{ddr}{dp^2} = 2A + 12Bp^2 + 30Cp^4 + 56Dp^6 + \&c.$$

$$\frac{ds}{dp} = 2\alpha p + 4\beta p^3 + 6\gamma p^5 + 8\delta p^7 + \&c.$$

$$\frac{dds}{dp^2} = 2\alpha + 12\beta p^2 + 30\gamma p^4 + 56\delta p^6 + \&c.$$

His valoribus ſubſtitutis prima æquatio abibit in hanc:

$$\left.\begin{array}{rrrrrr}
2\alpha p + 12\beta p^3 + & 30\gamma p^5 + & 56\delta p^7 + & 90\varepsilon p^9 + & \&c. & \\
- 2\alpha & - 12\beta & - 30\gamma & - 56\delta & - \&c. & \\
-2\alpha - 4\beta & - 6\gamma & - 8\delta & - 10\varepsilon & - \&c. & \\
- 2\alpha & - 4\beta & - 6\gamma & - 8\delta & - \&c. & \\
+ \alpha & + \beta & + \gamma & + \delta & + \&c. &
\end{array}\right\} = 0$$

§. XLVI. Si jam ſingularum poteſtatum ipſius p coefficientes nihilo æquales ponantur, erit:

$2\alpha - 2\alpha = 0$; α manet indeterminatum

$8\beta - 3\alpha = 0$; $\beta = \frac{1.3}{2.4}\alpha$

$24\gamma - 15\beta = 0$; $\gamma = \frac{3.5}{4.6}\beta = \frac{1.3.3.5}{2.4.4.6}\alpha$

$48\delta - 35\gamma = 0$; $\delta = \frac{5.7}{6.8}\gamma = \frac{1.3.3.5.5.7}{2.4.4.6.6.8}\alpha$

$80\varepsilon - 63\delta = 0$; $\varepsilon = \frac{7.9}{8.10}\delta = \frac{1.3.3.5.5.7.7.9}{2.4.4.6.6.8.8.10}\alpha$

&c. &c.

Si igitur valor coefficientis primi α conſtaret, quem quidem jam vidimus eſſe $= \frac{1}{2}$, omnes ſequentes coefficientes β, γ, δ, &c. forent cogniti. Verum reſolutio alterius æquationis quoque hunc nobis valorem ipſius α patefaciet.

§. XLVII. Subſtitutis enim ſeriebus ante traditis in altera æquatione proveniet;

$$\left.\begin{array}{rrrrrrr}
2Ap & +12Bp^3 & +30Cp^5 & +56Dp^7 & +90Ep^9 & +\&c. \\
 & -2A & -12B & -30C & -56D & -\&c. \\
-2A & -4B & -6C & -8D & -10E & -\&c. \\
 & -2A & -4B & -6C & -8D & -\&c. \\
+1 & +A & +B & +C & +D & +\&c. \\
-4\alpha & -8\beta & -12\gamma & -16\delta & -20\varepsilon & -\&c. \\
 & +4\alpha & +8\beta & +12\gamma & +16\delta & +\&c. \\
+2\alpha & +2\beta & +2\gamma & +2\delta & +2\varepsilon & +\&c.
\end{array}\right\} = 0$$

Unde

Unde ſimili modo elicitur:

$$2A - 2A + 1 - 2\alpha = 0; \text{ hinc fit } \alpha = \tfrac{1}{2}$$

$$8B - 3A - 6\beta + 4\alpha = 0; \quad 2.4\,B - 1.3\,A + 2\left(2 - \frac{1.3.3}{2.4}\right)\alpha = 0$$

$$24C - 15B - 10\gamma + 8\beta = 0; \quad 4.6\,C - 3.5\,B + 2\left(4 - \frac{3.5.5}{4.6}\right)\beta = 0$$

$$48D - 35C - 14\delta + 12\gamma = 0; \quad 6.8\,D - 5.7\,C + 2\left(6 - \frac{5.7.7}{6.8}\right)\gamma = 0$$

$$80E - 63D - 18\varepsilon + 16\delta = 0; \quad 8.10\,E - 7.9\,D + 2\left(8 - \frac{7.9.9}{8.10}\right)\delta = 0$$

§. XLVIII. Cognito igitur valore ipſius $\alpha = \frac{1}{2}$ altera ſeries s, quæ logarithmum ipſius p involvit, tota innoteſcit, erit enim:

$$\alpha = \tfrac{1}{2}$$

$$\beta = \frac{1.1.3}{2.2.4}$$

$$\gamma = \frac{1.1.3.3.5}{2.2.4.4.6}$$

$$\delta = \frac{1.1.3.3.5.5.7}{2.2.4.4.6.6.8}$$

$$\varepsilon = \frac{1.1.3.3.5.5.7.7.9}{2.2.4.4.6.6.8.8.10}$$

&c.

fietque hinc $s = \alpha pp + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p^{10} + \&c.$

§. XLIX. Quod autem ad alteram ſeriem attinet

$$r = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + \&c.$$

primus

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primus coefficiens A hinc manet indeterminatus, cujus rei ratio eſt, quod has ſeries ex æquatione differentiali ſecundi gradus elicuimus, quæ duplici determinatione indiget, ut ad noſtrum caſum accommodetur. Quare valorem hujus coefficientis A ex ipſa curvæ natura definiri oportet, eo autem invento, reliqui innoteſcent ex his formulis, ad quas ſuperiores redeunt.

$$B = \frac{1.3}{2.4} A - \frac{1}{8} \alpha \left(\frac{3}{2.2} + \frac{1}{1.1} \right)$$

$$C = \frac{3.5}{2.4} B - \frac{1}{8} \beta \left(\frac{3}{3.3} + \frac{1}{2.2} \right)$$

$$D = \frac{5.7}{6.8} C - \frac{1}{8} \gamma \left(\frac{3}{4.4} + \frac{1}{4.4} \right)$$

$$E = \frac{7.9}{8.10} D - \frac{1}{8} \delta \left(\frac{3}{5.5} + \frac{1}{4.4} \right)$$

§. L. His autem omnibus coefficientibus inventis ad datam quamvis abſciſſam CP $= p$, valor reſpondentis applicatæ PQ $= q$ ita definitur, ut ſit

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \&c.$$
$$- \alpha pplp - \beta p^4 lp - \gamma p^6 lp - \delta p^8 lp - \&c.$$

quæ ſeries ſi abſciſſa p fuerit unitate multo minor, ſatis promte convergit, ut inde valor ipſius q cognoſci queat. Hinc vero etiam applicatæ, quæ abſciſſis multo majoribus unitate reſpondent, definiri poterunt, quia abſciſſæ $\frac{1}{p}$ reſpondet applicata $\frac{q}{p}$. Quare ſi abſciſſa unitate multo major ponatur $=$ P eique reſpondens applicata $=$ Q ob $p = \frac{1}{P}$ & $q = p\,Q = \frac{Q}{P}$ erit

Q=

$$Q = P + A\overline{P}^{1} + B\overline{P}^{3} + C\overline{P}^{5} + D\overline{P}^{7} + \&c.$$
$$+ \alpha\overline{P}^{1}\, lP + \beta\overline{P}^{3}\, lP + \gamma\overline{P}^{5}\, lP + \delta\overline{P}^{7}\, lP + \&c.$$

Hinc ſi abſciſſa P fiat infinita erit

$$Q = P + \frac{\alpha lP}{P} \text{ ſeu } Q - P = \frac{\alpha lP}{P}$$

unde natura rami Dq in infinitum extenſi & ad aſymtotam CV appropinquantis colligitur.

§. LI. Quia porro novimus, ſi $p = 1$ fore $q = \frac{\pi}{2}$ pro hoc caſu æquatio inventa hanc formam ob $l1 =$ induet

$$\frac{\pi}{2} = 1 + A + B + C + D + E + \&c.$$

Cum igitur valor A nondum ſit definitus, reliqui vero B, C, D &c. ab eo pendeant, hæc æquatio conditionem continet, qua valor ipſius A determinatur. Ita ſcilicet valorem ipſius A comparatum eſſe oportet, ut ſumma ſeriei infinitæ $1 + A + B + C + \&c.$ fiat $\frac{\pi}{2}$. Verum ſi valores reliquarum litterarum B, C, D &c. qui ab A pendent, evolvantur, tam complicatæ reſultant expreſſiones, ut hinc valor ipſius A neutiquam erui poſſit.

§. LII. Ad hanc conſtantem A determinandam alia patet via, ſi datæ cujuſpiam ellipſis perimeter ex altera formula in numeris fuerit inventa. Quæ methodus cum requirat, ut omnes coefficientes in fractionibus decimalibus evolvantur, computo peracto repererietur:

$\alpha =$

$\alpha = 0,5000000000$	A quæritur	
$\beta = 0,1875000000;$	$B = 0,3750000000$	$A - 0,1093750000$
$\gamma = 0,1171875000;$	$C = 0,2343750000$	$A - 0,0820312500$
$\delta = 0,0854492188;$	$D = 0,1708984375$	$A - 0,0641886393$
$\varepsilon = 0,0672912598;$	$E = 0,1345825195$	$A - 0,0524978638$
$\zeta = 0,0555152893;$	$F = 0,1110305786$	$A - 0,0443481445$
$\eta = 0,0472540855;$	$G = 0,0945081711$	$A - 0,0383663416$
$\theta = 0,0411363691;$	$H = 0,0822727382$	$A - 0,0337966962$
$\iota = 0,0364228268;$	$I = 0,0728456536$	$A - 0,0301949487$
$\kappa = 0,0326793696;$	$K = 0,0653587392$	$A - 0,0272843726$

&c.

Hisque valoribus inventis, ſi abciſſa ſit $CP = p$, valor applicatæ q ita definietur ut ſit:

$$q = 1 + Ap^{2} + Bp^{4} + Cp^{6} + Dp^{8} + Ep^{10} + Fp^{12} + Gp^{14} + Hp^{16} + Ip^{18} + Kp^{20} + \&c.$$

$$- pplp.(\alpha + \beta p^{2} + \gamma p^{4} + \delta p^{6} + \varepsilon p^{8} + \zeta p^{10} + \eta p^{12} + \theta p^{14} + \iota p^{16} + \kappa p^{18} + \&c.)$$

§. LIII. Deinde vero ſupra ejusdem applicatæ q valorem ita invenimus expreſſum ut ſit:

$$q = \frac{\pi}{2}\left(1 - \frac{1.1}{2.2}(1-pp) - \frac{1.1.1.3}{2.2.4.4}(1-pp)^{2} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6}(1-pp)^{3} - \&c.\right)$$

Nunc igitur ex utraque formula pro eodem quopiam valo-

re ipſius p eruamus valorem ipſius q, ut deinceps ex æqualitate horum duorum elicere queamus valorem coefficientis A. Pro p vero non nimis exiguam fractionem ſubſtitui conveniet, ne expreſſio poſterior nimis lente convergat, tam parvum tamen aſſumamus ut coefficientes pro ſuperiore forma computati valori q ad 10 figuras inveniendo ſufficiant.

§. LIV. Ponamus ergo ad commodum calculi $p = \frac{1}{5}$. erit in logarithmis hyperbolicis:

$$-lp = 1,60943791243$$

Jam vero eſt

αpp	=	0,02000000000	
βp^4	=	0,00030000000	
γp^6	=	0,00000750000	
δp^8	=	0,00000021875	
εp^{10}	=	0,00000000689	
ζp^{12}	=	0,00000000023	
ηp^{14}	=	0,00000000001	
		0,02030772588	coefficiens ipſius $-lp$
		1,60943791243	
		0,03268402394	productum.

Deinde eſt

Ap

$Ap^2 =$	0,04000000000 A		
$Bp^4 =$	0,00060000000 A	—	0,00017500000
$Cp^6 =$	0,00001500000 A	—	0,00000525000
$Dp^8 =$	0,00000043750 A	—	0,00000016432
$Ep^{10} =$	0,00000001378 A	—	0,00000000538
$Fp^{12} =$	0,00000000045 A	—	0,00000000016
$Gp^{14} =$	0,00000000002 A	—	0,00000000001
	0,04061545175 A	—	0,00018041987

Ex his conficitur

$$q = 0,04061545175\,A + 1,032503250360407$$

§. LV. Nunc eundem valorem ipſius q ex altera æquatione quæramus, & cum ſit $p = \frac{1}{5}$, erit $1 - pp = \frac{24}{25}$ ſit $nn = \frac{24}{25}$ erit

$$q = \frac{\pi}{1}\left(1 - \frac{1.1}{2.2}nn - \frac{1.1.1.3}{2.2.4.4}n^4 - \frac{1.1.1.3.3.5}{2.2.4.4.6.6}n^6 - \&c.\right)$$

ponatur ad abbreviandum

$$q = \frac{\pi}{2} - \mathfrak{A}n^2 - \mathfrak{B}n^4 - \mathfrak{C}n^6 - \mathfrak{D}n^8 - \mathfrak{E}n^{10} - \&c.$$

Verum hoc caſu ob $nn = \frac{24}{25}$ ſeries iſta nimis lente convergit, quam ut hinc valor ipſius q ſatis exacte elici queat, quare ut utrinque parem convergentiam obtineamus ponamus $p = \frac{1}{\sqrt{2}}$, ut ſit tam $pp = \frac{1}{2}$ quam $nn = \frac{1}{2}$; calculum vero tantum ad 6 figuras expediamus: eritque

App

$App = 0,\ 500000\ A$
$Bp^4 = 0,\ 093750\ A - 0,027344$
$Cp^6 = 0,\ 029297\ A - 0,010254$
$Dp^8 = 0,\ 010681\ A - 0,004012$
$Ep^{10} = 0,004206\ A - 0,001640$
$Fp^{12} = 0,001735\ A - 0,000693$
$Gp^{14} = 0,000738\ A - 0,000300$
$Hp^{16} = 0,000321\ A - 0,000132$
$Ip^{18} = 0,000142\ A - 0,000059$
$Kp^{20} = 0,000064\ A - 0,000026$
Summa reliq: 60 A — 24

Sa. om. $0,640994\ A - 0,044484 + 0,\ 320497.\ l\frac{1}{p} + 1$

ergo $q = 1,066592 + 0,\ 640994\ A$
at altera expreſſio dat $q = 1,\ 350647$, unde ſit

$$A = \frac{284055}{640994} = 0,\ 443147$$

§. LVI. Quanquam hic valor non ultra 6 figuras extenditur, tamen caſui non tribuendum videtur, quod iſte numerus inventus 0, 443147 a logarithmo binarii 0, 69314718 unitatis quadrante 0, 25 præciſe deficiat. Quæ conjectura ſi veritati eſſet conſentanea, valorem litteræ A ad plurimas figuras exhibere liceret, cum enim ſit

$l2 = 0,\ 6931471805599453094172321$

foret $A = l2 - \frac{1}{4}$ ideoque

$A = 0,\ 4431471805599453094172321.$

Quod

Quod autem valor coefficientis hujus A ſit revera $= 12 - \frac{1}{4}$, ſequenti modo demonſtro, hancque conjecturam confirmo.

§. LVII. Comparo ſcilicet arcum ellipticum AYP, cujus ſemiaxes $AC = 1$, $CP = p$ cum arcu parabolico AZS ſuper eodem axe AC deſcripto, qui in A cum ellipſi communem habeat curvaturam. Sumta abſciſſa communi $AX = x$, ſit applicata ellipſis $XY = y$ & parabolæ $XZ = z$, erit $y = p\sqrt{(2x - xx)}$ & $z = p\sqrt{2x}$, ideoque $dy = \frac{pdx(1-x)}{\sqrt{(2x-xx)}}$ & $dz = \frac{pdx}{\sqrt{2x}}$: unde ſit Fig. 2.

arcus ellipticus $AY = \int dx\sqrt{(1 + \frac{pp(1-x)^2}{2x - xx})}$

arcus parabolicus $AZ = \int dx\sqrt{(1 + \frac{pp}{2x})}$. Conſtat autem

eſſe $AZ = x\sqrt{(1 + \frac{pp}{2x})} + \frac{1}{4}pp\,l\frac{\sqrt{(1+\frac{pp}{2x})}+1}{\sqrt{(1+\frac{pp}{2x})}-1}$; Hinc ſi ponatur

$x = 1$, erit arcus parabolicus $AZS = \sqrt{(1 + \frac{1}{2}pp)} + \frac{1}{4}pp\,l\frac{\sqrt{(1+\frac{1}{2}pp)}+1}{\sqrt{(1+\frac{1}{2}pp)}-1}$.

At in formulis integralibus erit:

$$\sqrt{(1 + \frac{pp(1-x)^2}{2x - xx})} = \sqrt{(1 + \frac{pp}{2x} - \frac{pp(3-2x)}{4-2x})}$$

Quia autem comparationem non ad altiores ipſius p poteſtates extendere opus eſt quam ad ſecundam: coëfficientes enim altiorum ipſius p poteſtatum ex minoribus jam definivimus, rejectis terminis, qui continent p^4 & altiores poteſtates, erit:

 $\sqrt{(1}$

$\sqrt{(1+\frac{pp(1-x)^2}{2x-xx})} = \sqrt{(1+\frac{pp}{2x}-\frac{pp(3-2x)}{2(2-x)})}$ ideoque

$AY = \int dx\sqrt{(1+\frac{pp}{2x})} - \frac{1}{4}pp\int\frac{dx(3-2x)}{2-x}$, integralibusque actu ſumtis

$$AY = x\sqrt{(1+\frac{pp}{2x})} + \frac{1}{4}ppl\,\frac{\sqrt{(1+\frac{pp}{2x})}+1}{\sqrt{(1+\frac{pp}{2x})}-1} - \frac{1}{2}ppx - \frac{1}{4}ppl\,\frac{2-x}{2}$$

Ponatur jam $x = 1$, ut prodeat arcus $AYP = q$, erit

$q = \sqrt{(1+\frac{1}{2}pp)} + \frac{1}{4}ppl(\sqrt{(1+\frac{1}{2}pp)}+1) - \frac{1}{4}ppl(\sqrt{(1+\frac{1}{2}pp)}-1) - \frac{1}{2}pp + \frac{1}{4}ppl2.$

§. LVIII. Jam quoniam ad altiores ipſius p poteſtates non reſpicimus, erit $\sqrt{(1+\frac{1}{2}pp)} = 1+\frac{1}{4}pp$, unde fiet

$$q = 1+\frac{1}{4}pp+\frac{1}{4}pp\,l(2+\frac{1}{4}pp) - \frac{1}{4}pp\,l\frac{1}{4}pp - \frac{1}{2}pp + \frac{1}{4}ppl2$$

ubi pro $l(2+\frac{1}{4}pp) = l2+\frac{1}{8}pp$ ſcribere licet $l2$, ita ut

ſit $q = 1-\frac{1}{4}pp+\frac{1}{2}ppl2-\frac{1}{2}pplp+\frac{1}{2}ppl2$

ſeu $q = 1-\frac{1}{2}pplp+pp(l2-\frac{1}{4})$

unde perſpicitur coefficientem ipſius pp, quem ante litera A indica-

vimus

vimus eſſe $= l2 - \frac{1}{4}$; omnino uti ex caſu ante computato conjectura ſumus conſecuti.

§. LIX. Pro curva igitur initio propoſita AQDq, ſi fuerit abſciſſa CP $= p$ & applicata PQ $= q$, erit Fig. I.

$$q = 1 + App + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + \&c.$$
$$- (\alpha pp + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p^{10} + \&c.)\, lp$$

ubi coefficientes ita determinantur:

$A = l2 - \frac{1}{4}$	$\alpha = \frac{1}{2}$
$B = \frac{1.3}{2.4} A - \frac{1}{2}(\alpha - \beta) + \frac{1}{2} \cdot \frac{\beta}{2}$	$\beta = \frac{1.3}{2.4}\, \alpha$
$C = \frac{3.5}{4.6} B - \frac{1}{3}(\beta - \gamma) + \frac{1}{4} \cdot \frac{\gamma}{3}$	$\gamma = \frac{3.5}{4.6}\, \beta$
$D = \frac{5.7}{6.8} C - \frac{1}{4}(\gamma - \delta) + \frac{1}{6} \cdot \frac{\delta}{4}$	$\delta = \frac{5.7}{6.8}\, \gamma$
$E = \frac{7.9}{8.10} D - \frac{1}{5}(\delta - \varepsilon) + \frac{1}{8} \cdot \frac{\varepsilon}{5}$	$\varepsilon = \frac{7.9}{8.10}\, \delta$
$F = \frac{9.11}{10.12} E - \frac{1}{6}(\varepsilon - \zeta) + \frac{1}{10} \cdot \frac{\zeta}{6}$	$\zeta = \frac{9.11}{10.12}\, \varepsilon$

&c.

ſeries hæc valde convergit, ſi abſciſſa p fuerit fractio valde parva, ſin autem ſit unitate multo major, iisdem manentibus coefficientibus erit

$q =$

$$q = p + \frac{A}{p} + \frac{B}{p^3} + \frac{C}{p^5} + \frac{D}{p^7} + \frac{E}{p^9} + \&c.$$
$$+ \left(\frac{\alpha}{p} + \frac{\beta}{p^3} + \frac{\gamma}{p^5} + \frac{\delta}{p^7} + \frac{\varepsilon}{p^9} + \&c. \right) lp$$

§. LX. Verum ſi abſciſſa p non multum ab unitate diſcrepet, uti conveniet hac ſerie ſupra §. XXVI. inventa

$$q = 1 + pp \left\{ \begin{array}{l} \left(\frac{\pi}{2} - 1\right) + \left(\frac{1.3}{2.2}.\frac{\pi}{2} - 1\right)(1-pp) + \\ \left(\frac{1.3.3.5}{2.2.4.4}.\frac{\pi}{2} - 1\right)(1-pp)^2 + \left(\frac{1.3.3.5.5.7}{2.2.4.4.6.6}\right. \\ \left.\frac{\pi}{2} - 1\right)(1-pp)^3 + \&c. \end{array} \right.$$

quæ etiam ex natura ellipſis in hanc convertitur

$$q = p + \frac{1}{p} \left\{ \begin{array}{l} \left(\frac{\pi}{2} - 1\right) - \left(\frac{1.3}{2.2}.\frac{\pi}{2} - 1\right)\frac{(1-pp)}{pp} + \\ \left(\frac{1.3.3.5}{2.2.4.4}\frac{\pi}{2} - 1\right)\frac{(1-pp)^2}{p^4} - \left(\frac{1.3.3.5.5.7}{2.2.4.4.6.6}\right. \\ \left.\frac{\pi}{2} - 1\right)\frac{(1-pp)^3}{p^6} + \&c. \end{array} \right)$$

unde prout fuerit vel $p > 1$ vel $p < 1$ eam eligere licet, cujus termini vel iisdem ſignis procedant, vel alternantibus. Plerumque autem præſtat ad ſummam proxime definiendam ſigna eligere alternantia.

Problema.

§. LXI. Datis axibus conjugatis ellipſis, in numeris proxime exhibere ejus perimetrum.

Solutio.

Sint ſemiaxes ellipſis 1 & p, & quadrans perimetri $= q$, atque per formulas inventas valor ipſius q in numeris definiri poterit, dummodo ea eligatur, cujus termini maxime convergant. Quatuor autem adepti ſumus formulas quæ ſunt:

I. $q = 1 + App + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + Fp^{12}$ &c.

$- (\alpha pp + \beta p^4 + \gamma p^6 + \delta p^8 + \epsilon p^{10} + \zeta p^{12}$, &c.$)\ lp$

II. $q = p + A\frac{1}{p} + B\frac{1}{p^3} + C\frac{1}{p^5} + D\frac{1}{p^7} + E\frac{1}{p^9} + F\frac{1}{p^{11}}$ + &c.

$+ (\frac{\alpha}{p} + \frac{\beta}{p^3} + \frac{\gamma}{p^5} + \frac{\delta}{p^7} + \frac{\epsilon}{p^9} + \frac{\zeta}{p^{11}}$ + &c.$)\ lp$

III. $q = 1 + pp\,(\mathfrak{A} + \mathfrak{B}(1-pp) + \mathfrak{C}(1-pp)^2 + \mathfrak{D}(1-pp)^3 + \mathfrak{E}(1-pp)^4$ + &c.$)$

IV. $q = p + \frac{1}{p}(\mathfrak{A} - \mathfrak{B}\frac{(1-pp)}{pp} + \mathfrak{C}\frac{(1-pp)^2}{p^4} - \mathfrak{D}\frac{(1-pp)^3}{p^6} + \mathfrak{E}\frac{(1-pp)^4}{p^8}$ − &c.$)$

Horum

Horum autem tergeminorum coefficientium valores ſunt in numeris:

A = 0,44314718056	α = 0,50000000000	𝔄 = 0,57079632679
B = 0,05680519271	β = 0,18750000000	𝔅 = 0,17809724510
C = 0,02183137044	γ = 0,11718750000	ℭ = 0,10446616728
D = 0,01154452143	δ = 0,08544921875	𝔇 = 0,07378655152
E = 0,00714200029	ε = 0,06729125977	𝔈 = 0,05700863665
F = 0,00485474337	ζ = 0,05551527931	𝔉 = 0,04643855029
G = 0,00351468795	η = 0,04725408554	𝔊 = 0,03917161591
H = 0,00266223578	θ = 0,04113636911	ℌ = 0,03386971991
I = 0,00208639732	ι = 0,03642282682	ℑ = 0,02983116632
K = 0,00167916842	$\varkappa$ = 0,03267936962	𝔎 = 0,02665267507
		𝔏 = 0,02408604339

Hinc pro quavis ellipſis ſpecie habebitur ſeries convergens, unde ejus perimeter definiri poterit, veluti

ſi ponatur $p = \frac{1}{10}$ erit $q = 1,0159935021$

ſi ſit $p = \frac{1}{5}$ erit $q = 1,0505022700$

ſi ſit $p = \frac{1}{\sqrt{2}}$ erit $q = 1,3506429$

Tabula

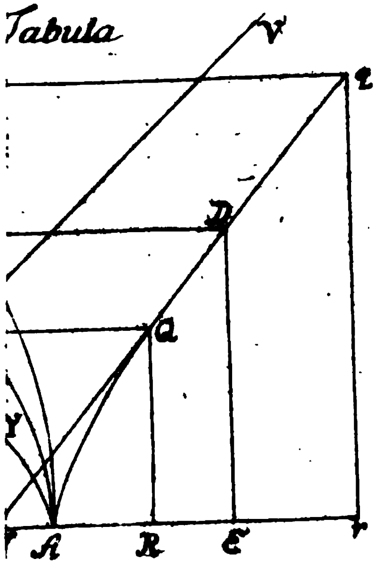


Fig. 1.

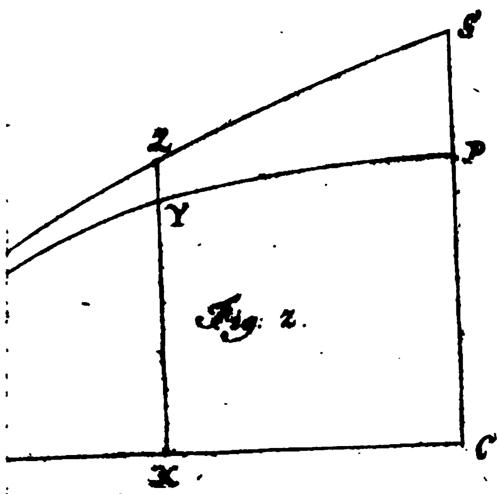


Fig. 2.

…ad. F.H.Frisch

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